Lecture 9: Alexander Duality

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Let X be a finite CW complex embedded cellularly into S^n, i.e. $X \hookrightarrow S^n$. Let $A \subset S^n - X$ be a finite CW complex such that A is homotopy equivalent to $S^n - X$.

Example 1.1. $X = S^{p-1}$, $A = S^{q-1}$, $n = p + q - 1$.

The *join* of X and A is $X \star A = X \times [0,1] \times A/\sim$ where $(x, 0, a_1) \sim (x, 0, a_2)$ and $(x_1, 1, a) \sim (x_2, 1, a)$. It can be expressed as the colimit or push-out

$$\begin{array}{ccc} X \times A & \longrightarrow & \mathcal{C}X \times A \\ \downarrow & & \downarrow \\ X \times \mathcal{C}A & \longrightarrow & X \star A \end{array},$$

where $\mathcal{C}X$ denotes the non-reduced cone on X. (Push-out means a colimit of this shape.) There is a homotopy equivalence $X \star A \to \Sigma(X \wedge A)$. You can see this by hand, or alternatively note that the columns of the diagram

$$\begin{array}{ccccc} X & \longleftarrow & X \vee A & \longrightarrow & A \\ \downarrow{=} & & \downarrow & & \downarrow{=} \\ X & \longleftarrow & X \times A & \longrightarrow & A \\ \downarrow & & \downarrow & & \downarrow \\ * & \longleftarrow & X \wedge A & \longrightarrow & * \end{array}$$

are cofiber sequences. Take the homotopy push-out of all rows to obtain the cofiber sequence

$$* \to ? \to \Sigma(X \wedge A),$$

where ? is the push-out of

$$X \leftarrow X \times A \to A.$$

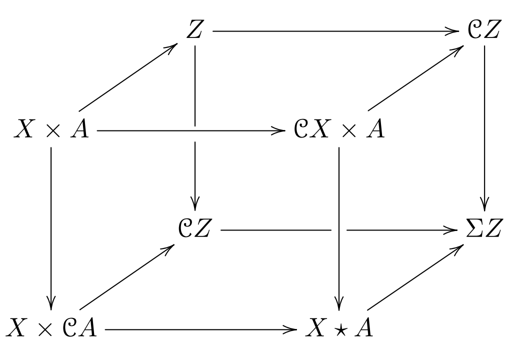
This gives that ? is $\Sigma(X \wedge A)$. On the other hand, ? is homotopy equivalent to the push-out of

$$X \times \mathcal{C}A \leftarrow X \times A \rightarrow \mathcal{C}X \times A,$$

which is $X \star A$.

Assume for the moment that no point of X is antipodal to a point of A. Then we have a map $X \star A \to S^n$ given by $(x, t, a) \mapsto \gamma_{x,a}(t)$, where $\gamma_{x,a} : [0,1] \to S^n$ is the geodesic of length one from x to a. Thus we have a map $\Sigma X \wedge A \to S^n$. In the stable homotopy category, we therefore have a map $\Sigma^{-(n-1)} A \wedge X \to S^0$. It follows that we have a natural map $\Sigma^{-(n-1)} A \to F(X, S^0) = DX$.

If there is a pair of antipodal points, one in X and one in A, we still have an analogous map $\Sigma^{-(n-1)} A \to DX$. Namely, wiggle A so there is a point of S^n that is not in $X \cup A$. Fix a point p not in $X \cup A$. Let $\mathrm{St} : S^n - \{p\} \to \mathbb{R}^n$ denote the stereographic projection. We obtain a map $\mu : X \times A \to S^{n-1}$ by $\mu(x, a) = (\mathrm{St}(x) - \mathrm{St}(a))/|\mathrm{St}(x) - \mathrm{St}(a)|$. Given a map $X \times A \to Z$, the *Hopf construction* is the map $X \star A \to \Sigma Z$ obtained by the morphism of pushout diagrams from the push-out diagram formed by the front face to the push-out formed by the back face



There are nice pictures in [M, p. 54].

Applying the Hopf construction to to μ we obtain the desired map $X \star A \to \Sigma S^{n-1}$, whence $\Sigma^{-(n-1)} A \to DX$.

We will show that $\Sigma^{-(n-1)} A \to DX$ is an isomorphism in the stable homotopy category. We will use the following facts.

1. Let Z be a spectrum and let Y be a finite spectra. Then $DY \wedge Z \cong F(Y, Z)$.

   Proof. By the definition of $F(Y, Z)$, what we need to show is that for any spectrum X, we have $[X, DY \wedge Z] \cong [X \wedge Y, Z]$. This was Exercise 1.3 (1) and (2) from L8. □

2. $\pi_r F(Y,Z) = [Y,Z]_r$.

   Proof. By Lemma 1.2 of Lecture 4, which is [A, III Prop 2.8], we have $\pi_r F(Y,Z) \cong [\Sigma^\infty S^r, F(Y,Z)]$. By the definition of $F(Y,Z)$ we have $[\Sigma^\infty S^r, F(Y,Z)] \cong [S^r \wedge Y, Z]$. By our identification of desuspension with a shift, we also have suspension identified with a shift. It follows that $[S^r \wedge Y, Z] \cong [Y,Z]_r$. □

3. Let Y be a finite spectrum. Suppose $[Y, H\mathbb{Z}]_r = 0$ for all $r \in \mathbb{Z}$. Then $Y \cong *$.

   Proof. Choose n such that Y_n contains representatives for all the stable cells. There is a finite subcomplex $K \subset Y_n$ containing all the chosen representatives. By replacing K by its suspension $\Sigma K \subset Y_{n+1}$, we may assume that K is simply connected.This inclusion induces a map $\Sigma^{-n}K \to Y$ which is the inclusion of a cofinal subspectrum, and therefore an isomorphism in the stable homotopy category. Thus $0 = [\Sigma^{-n}K, H\mathbb{Z}]_r \cong [K, H\mathbb{Z}]_{r-n} \cong \tilde{\mathrm{H}}^{n-r}(K,\mathbb{Z})$ by Lecture 4. This implies that K is null-homotopic by the Hurewicz theorem and the universal coefficient theorem. □

4. If X is a CW complex, $\pi_r(X \wedge H\mathbb{Z}) \cong \tilde{H}_r(X,\mathbb{Z})$. We'll postpone the proof of this until we do generalized homology. For any spectrum E, the generalized reduced E-homology of X is $E_r X = \pi_r(X \wedge E)$.

Theorem 1.2. *(Alexander Dualty) Let X be a finite CW complex embedded into S^n, such that $S^n - X$ is homotopy equivalent to a finite CW complex. Then*

$$DX \cong \Sigma^{-(n-1)}(S^n - X)$$

is an isomorphism in the stable homotopy category. Furthermore, for any spectrum E,

$$E^r X \cong E_{n-r-1}(S^n - X) \tag{1}$$

$$E_r X \cong E^{n-1-r}(S^n - X). \tag{2}$$

For example, $\tilde{\mathrm{H}}_r(X,\mathbb{Z}) \to \tilde{\mathrm{H}}^{n-1-r}(S^n - X,\mathbb{Z})$ is an isomorphism.

The assertion $DX \cong \Sigma^{-(n-1)}(S^n - X)$ implies the other assertions of the theorem as follows. We have a map $DX \to \Sigma^{-(n-1)}(S^n - X)$, which we are assuming to be an isomorphism. Smashing with E and taking π_r shows (1). Changing the roles of $S^n - X$ and X, as well as changing r to $n-1-r$, shows (2).

To prove Theorem 1.2, we first show that the converse of this implication holds as well.

Proposition 1.3. *If the induced map* $\tilde{\mathrm{H}}_r(X,\mathbb{Z}) \to \tilde{\mathrm{H}}^{n-1-r}(S^n - X, \mathbb{Z})$ *is an isomorphism, then* $\Sigma^{-(n-1)}A \to F(X, S^0)$ *is an isomorphism in the stable homotopy category.*

Proof. It suffices to show that this map is an isomorphism on π_r. Since a cofiber sequence induces a long exact sequence in homotopy groups, it suffices to show that the cofiber $Y = F(X,S^0) \cup C(\Sigma^{-(n-1)}A)$ of $\Sigma^{-(n-1)}A \to F(X,S^0)$ satisfies $\pi_r Y = 0$. By (3), we may show $[Y, H\mathbb{Z}]_r = 0$. Since

$$\Sigma^{-(n-1)}A \to F(X,S^0) \to Y$$

is a cofiber sequence, we have that

$$DY \to X \to D(\Sigma^{-(n-1)}A)$$

is a cofiber sequence. Since smashing with $H\mathbb{Z}$ preserves cofiber sequences, we have that

$$DY \wedge H\mathbb{Z} \to X \wedge H\mathbb{Z} \to D(\Sigma^{-(n-1)}A) \wedge H\mathbb{Z}$$

is a cofiber sequence. By (1), $DY \wedge H\mathbb{Z} = F(Y, H\mathbb{Z})$. By (2), we have that $\pi_r F(Y, H\mathbb{Z}) = [Y, H\mathbb{Z}]_r$. Thus it suffices to show that $X \wedge H\mathbb{Z} \to D(\Sigma^{-(n-1)}A) \wedge H\mathbb{Z}$ induces an isomorphism on π_r for all r. By (4), $\pi_r(X \wedge H\mathbb{Z}) \cong \tilde{\mathrm{H}}_r(X,\mathbb{Z})$. By (1) and (2) and Prop 1.1 Lecture 4, we have $\pi_r D(\Sigma^{-(n-1)}A) \wedge H\mathbb{Z}) \cong \tilde{\mathrm{H}}^{n-1-r}(A,\mathbb{Z})$. Thus what we have to show is that the constructed natural map

$$\tilde{\mathrm{H}}_r(X,\mathbb{Z}) \to \tilde{\mathrm{H}}^{n-1-r}(A,\mathbb{Z})$$

is an isomorphism, which is our hypothesis. ☐

The assertion $\tilde{\mathrm{H}}_r(X,\mathbb{Z}) \to \tilde{\mathrm{H}}^{n-1-r}(A,\mathbb{Z})$ is also called Alexander duality, which is Proposition 1.4, or [H, Theorem 3.44].

Proposition 1.4. $\mathrm{H}_r(X,\mathbb{Z}) \to \mathrm{H}^{n-1-r}(A,\mathbb{Z})$ *is an isomorphism.*

Proof. We've seen that Proposition 1.4 holds if and only if $\Sigma^{-(n-1)}A \to F(X,S^0)$ is an isomorphism in the stable homotopy category. We show both by induction on the number of cells of X. If X has one cell, X is a point and the claim holds. Since gluing on a cell can be expressed as a cofiber sequence with a sphere, let's also do the case where X is a sphere. If X is embedded as a lower dimensional equator, then we can see directly that the complement $A \cong S^n - X$ is as in Example 1.1. However, part of what we need to show is that the claims hold for all the embeddings. We proved [H, Proposition 2B.1] last semester in Math 6441, which computes the homology of $S^n - S^k$ for any embedding, and this shows the claim for a sphere. We should really be showing that all these maps are compatible, so here is another way of doing the case where X is a sphere.

If $H : X \times [0,1] \to S^n$ is an isotopy between $f : X \to S^n$ and $g : X \to S^n$, then the maps

$$S^n - f(X) \to S^n \times [0,1] - H(X \times [0,1]) \to S^n \times [0,1] - f(X) \to S^n - f(X)$$

define a homotopy equivalence between $S^n - f(X)$ and $S^n \times [0,1] - H(X \times [0,1])$. Running the same argument with g replacing f, we have that $S^n - f(X)$ is homotopy equivalent to $S^n - g(X)$. Now note that if $X \subset S^n$ is an embedding, and if we then embed $S^n \subset S^{n+1}$ along the equator, we get a new embedding $X \subset S^{n+1}$ such that $S^{n+1} - X \cong \Sigma(S^n - X)$. If we prove that $DX \cong \Sigma^{-((n+1)-1)}(S^{n+1} - X)$, we then have $DX \cong \Sigma^{-((n+1)-1)}\Sigma(S^n - X) \cong \Sigma^{-(n-1)}(S^n - X)$, which is what we wanted to show. In other words, given an embedding $X \subset S^n$, we may assume that n is as large as we wish. For n sufficiently large, the Whitney embedding theorem says that all embeddings are isotopic, so we may use Example 1.1 to prove the general case when X is a sphere. So, we have the case where X has one cell, and the case where X is a sphere.

For the inductive step, it remains to show that if $B \to X \to Y$ is a cofiber sequence such that B and X satisfy Proposition 1.4 (for any embedding satisfying the hypotheses), then Y satisfies Proposition 1.4. If $B \subset S^m$ and $X \subset S^p$, then the mapping cylinder of $B \to X$ is embedded inside $S^m \star S^p \cong S^{m+p+1}$ (see [A, p. 191]). Thus we may assume that $B \subset X$ are embedded in a single sphere S^n. Since taking duals preserves cofiber sequences,

$$DY \to DX \to DB$$

is a cofiber sequence. By the inductive hypothesis, $DX \cong \Sigma^{-(n-1)}(S^n - X)$ and $DB \cong \Sigma^{-(n-1)}(S^n - B)$. So we are reduced to showing that

$$\Sigma^{-(m-1)}(S^m - Y) \to \Sigma^{-(n-1)}(S^n - X) \to \Sigma^{-(n-1)}(S^n - B)$$

is a cofiber sequence for Y embedded in some S^m. Embedding S^n into S^{n+1} along the equator, and choose $m = n+1$. Then the cone on B can be moved into the north pole of S^{n+1}, giving an embedding of Y. From the picture, we see that

$$(S^n - X) \to (S^n - B) \to (S^{n+1} - Y)$$

is a cofiber sequence. Note that this also shows that any finite CW complex can be embedded in some sphere.

Now, we really want to start with $Y \subset S^m$ for some m satisfying the hypothesis and show the claim for this embedding. So, choose a top cell D of Y, and express Y as a cofiber sequence $S^d = B = \partial D \to X \to Y$ all embedded in S^m. We are assuming $Y \subset S^m$. We embed S^m as the equator in S^{m+1}. Then the complement of Y in S^{m+1} is $S^{m+1} - Y = \Sigma(S^m - Y)$. So showing the claim for this embedding in S^{m+1} shows the claim for the embedding we started with.

We claim that this equatorial embedding of Y has equivalent complement to the embedding of Y we just constructed associated to $B \to X$ in S^m. Both embeddings are formed by filling in an embedded disk to the same embedded circle in S^{m+1}, and by moving the equatorial embedding into the upper hemisphere, we get a family of embeddings, giving a homotopy equivalence of the complements. □

References

[A] J.F. Adams, *Stable Homotopy and Generalized Homology* Chicago Lectures in Mathematics, The University of Chicago Press, 1974.

[H] Allen Hatcher, *Algebraic Topology.*

[M] Haynes Miller, *Vector Fields on spheres, etc. (course notes).*