Lecture 7: Cofiber sequences are fiber sequences

1/23/15

1 Cofiber sequences and the Puppe sequence

If $f : X \to Y$ is a map of CW complexes, recall that the reduced *mapping cone* is the space $Y \cup_f CX = (Y \coprod X \wedge [0,1])/\sim$, where $(x,1) \sim f(x)$. If we vary f by a homotopy, $Y \cup_f CX$ changes by a homotopy equivalence.

We may likewise form the reduced mapping cone of a map $f : X \to Y$ in the stable homotopy category. f is represented by a function of spectra $f : X' \to Y$, where X' is a cofinal subspectrum of X. By replacing f by a homotopic map, we may assume that $f'_n : X'_n \to Y_n$ is a cellular map. Then $Y \cup_f CX$ is the spectrum whose nth space is $Y_n \cup_{f_n} CX'_n$ and whose structure maps are induced from those of X and Y. This is well-defined up to isomorphism, because varying f by a homotopy does not change the isomorphism class of $Y \cup_f CX$.

If $i : A \to X$ is the inclusion of a closed subspectrum, then define X/A be the spectrum whose nth space is X_n/A_n and whose structure maps are those induced from the structure maps of X. The evident map $X \cup_i CA \to X/A$ is an isomorphism in the stable homotopy category because on the level of spaces we have homotopy equivalences which therefore induce isomorphism on π_*.

Definition 1.1. *A* cofiber sequence *is any sequence equivalent to a sequence of the form* $X \xrightarrow{f} Y \xrightarrow{i} Y \cup_f CX$

Proposition 1.2. *([A, III Prop 3.9]) For each Z, the sequence* $[Y \cup_f CX, Z] \to [Y, Z] \to [X, Z]$ *is exact.*

Proof. Since the composite $X \to Y \cup_f CX$ is null, we have that the image of $[Y \cup_f CX, Z] \to [Y, Z]$ is indeed in the kernel of $[Y, Z] \to [X, Z]$. Suppose $g : Y' \to Z$ is a function such that Y' is a cofinal subspectrum of Y and such that the associated morphism is null in $[X, Z]$. We wish to construct a pmap $Y \cup_f CX \to Z$ extending the pmap $g : Y \to Z$. To do this, choose a cofinal subspectrum X' of X such that gf is defined as a function on X' and such that there is a function $H : X' \wedge [0,1]_+ \to Z$ giving a homotopy between gf and the constant map. We checked that we may choose a cofinal subspectrum Y'' of Y containing the image of X' . H and g determine a function $Y'' \cup_f CX' \to Z$. $\square$

Any map can be extended to a cofiber sequence. In particular, we can extend cofiber sequences to the right

$$X \xrightarrow{f} Y \xrightarrow{i} Y \cup_f CX \to (Y \cup_f CX) \cup_i CY \to$$

Since $(Y \cup_f CX) \cup_i CY \cong \Sigma X$, we get that

$$X \xrightarrow{f} Y \xrightarrow{i} Y \cup_f CX \to \Sigma X \to \Sigma Y \to \Sigma(Y \cup_f CX) \to \Sigma\Sigma X \tag{1}$$

has all three term sequences cofiber sequences.

Note that desuspensions and suspensions of cofiber sequences are cofiber sequences. Applying Σ^{-1} to the cofiber sequence $Y \cup_f CX \to \Sigma X \to \Sigma Y$, we have that $\Sigma^{-1} Y \cup_f CX \to X \to Y$ is a cofiber sequence. Thus we may continue (1) to the left.

Corollary 1.3. *If $X \to Y \to Z$ is a cofiber sequence in the stable homotopy category, then for any W, the sequence*

$$\ldots \to [X,W]_{n+1} \to [Z,W]_n \to [Y,W]_n \to [X,W]_n \to \ldots$$

is exact.

Proof. The sequence

$$\ldots \to [\Sigma^{n+1}X,W] \to [\Sigma^n Z,W] \to [\Sigma^n Y,W] \to [\Sigma^n X,W] \to \ldots$$

is exact by the above chain of cofiber sequences and Proposition 1.2. Since we have identified desuspension with a shift, suspension may also be identified with a shift. Thus $[\Sigma^n Y, W] = [Y,W]_n$. $\square$

2 Fiber sequences are cofiber sequences

Proposition 2.1. *If $X \xrightarrow{f} Y \xrightarrow{i} Z$ is a cofiber sequence in the stable homotopy category, then for any W, the sequence*

$$\ldots \to [W,X]_n \to [W,Y]_n \to [W,Z]_n \to [W,X]_{n-1} \to \ldots$$

is exact.

Proof. As above, it suffices to show that

$$[W, X] \to [W, Y] \to [W, Z]$$

is exact. Since the composite $X \to Z$ is null, we have that the composite $[W, X] \to [W, Z]$ is 0. Let $g : W \to Y$ be a pmap such that ig is nullhomotopic. The choice of a null-homotopy gives the morphism $h : CW \to Z$ from cone on W to Z. We then obtain j and k in the commutative diagram

$$\begin{array}{ccccccccc}
X & \xrightarrow{f} & Y & \xrightarrow{i} & Z & \longrightarrow & \Sigma X & \xrightarrow{-f} & \Sigma Y \\
 & & \uparrow{\scriptstyle g} & & \uparrow{\scriptstyle h} & & \uparrow{\scriptstyle j} & & \uparrow{\scriptstyle \Sigma g} \\
W & \longrightarrow & W & \longrightarrow & CW & \longrightarrow & \Sigma W & \xrightarrow{-1} & \Sigma W
\end{array}$$

Since suspension is an equivalence, we have that the image of $\Sigma^{-1}j$ under $[W, X] \to [W, Y]$ is g. □

One could define a fiber sequence to be $X \to Y \to Z$ such that the composite $X \to Z$ was the constant map and such that the sequence satisfies the conclusion of Proposition 2.1. Dually, one could also define a cofiber sequence to be $X \to Y \to Z$ such that $X \to Z$ is null and satisfying the conclusion of 1.3. This is the same as the above (exercise: use a natural map and the 5 lemma to show it induces an isomorphism on π_*). Proposition 2.1 can therefore be stated by saying that fiber sequences and cofiber sequences are the same in the stable homotopy category.

References

[A] J.F. Adams, *Stable Homotopy and Generalized Homology* Chicago Lectures in Mathematics, The University of Chicago Press, 1974.