

# Lecture 6: Suspension is an equivalence

1/20/14

## 1 Suspension can be inverted

Let  $E$  be a CW spectrum.

**Definition 1.1.** The fake suspension  $\Sigma_f E$  of  $E$  is the CW spectrum with spaces  $(\Sigma E)_n = S^1 \wedge E_n$  and structure maps  $S^1 \wedge (\Sigma E)_n = S^1 \wedge S^1 \wedge E_n \xrightarrow{1 \wedge \epsilon_n} S^1 \wedge E_{n+1} = (\Sigma E)_{n+1}$ .

**Definition 1.2.** Let  $\Sigma^{-1} E$  be the CW spectrum with spaces  $(\Sigma^{-1} E)_n = E_{n-1}$  and  $n$ th structure map given by  $E$ 's  $n - 1$ st structure map  $\epsilon_{n-1}$ .

**Proposition 1.3.** There are natural isomorphisms in the stable homotopy category between

1.  $E$  and  $\Sigma^{-1} \Sigma_f E$ ,
2.  $E$  and  $\Sigma_f \Sigma^{-1} E$ .

*Proof.* The  $n$ th space of  $\Sigma_f \Sigma^{-1} E$  and  $\Sigma^{-1} \Sigma_f E$  are both  $\Sigma E_{n-1}$ . The structure maps of  $\Sigma_f \Sigma^{-1} E$  and  $\Sigma^{-1} \Sigma_f E$  can likewise be identified, showing that there is a canonical isomorphism  $\Sigma_f \Sigma^{-1} E \cong \Sigma^{-1} \Sigma_f E$ . Thus (1) and (2) are equivalent.

(1):  $\epsilon_{n-1} : \Sigma E_{n-1} \rightarrow E_n$  is the  $n$ th map in a function of spectra  $f : \Sigma^{-1} \Sigma_f E \rightarrow E$ . We claim that this map induces an isomorphism on homotopy groups, showing the claim by the last lecture.

$\pi_r \Sigma^{-1} \Sigma_f E = \text{colim}_{n \rightarrow \infty} \pi_{n+r} \Sigma E_{n-1}$ . Suspension induces a map  $\pi_{n+r-1} E_{n-1} \rightarrow \pi_{n+r} \Sigma E_{n-1}$ . These maps define a map  $\phi : \text{colim}_{n \rightarrow \infty} \pi_{n+r-1} E_{n-1} \rightarrow \text{colim}_{n \rightarrow \infty} \pi_{n+r} \Sigma E_{n-1}$ . Note that  $\phi$  is a map

$$\phi_r : \pi_r E \rightarrow \pi_r \Sigma^{-1} \Sigma_f E.$$

The map  $f$  induces  $\pi_r(f) : \pi_r \Sigma^{-1} \Sigma_f E \rightarrow \pi_r E$ .

It suffices to see that  $\pi_r(f)$  and  $\phi_r$  are inverses. This is a consequence of the algebraic fact that given a diagram

$$\begin{array}{ccccc} & & A_n & & A_{n+1} & \dots \\ & \nearrow & & \searrow & \nearrow & \\ \dots & & & & & \\ & B_n & & B_{n+1} & & \end{array}$$

the upward and downward facing arrows induce natural inverse isomorphisms

$$\operatorname{colim}_{n \rightarrow \infty} A_n \cong \operatorname{colim}_{n \rightarrow \infty} B_n.$$

□

When we define the smash product of spectra, the notion of suspension of  $E$  will be fixed. Namely, we will have that the suspension of  $E$  is  $\Sigma E = S^1 \wedge E$ . This suspension will have spaces  $S^1 \wedge E_n$  and structure maps

$$S^1 \wedge (\Sigma E)_n = S^1 \wedge S^1 \wedge E_n \xrightarrow{\tau \wedge 1} S^1 \wedge S^1 \wedge E_n \xrightarrow{1 \wedge \epsilon_n} S^1 \wedge E_{n+1} = (\Sigma E)_{n+1},$$

where  $\tau : S^1 \wedge S^1 \rightarrow S^1 \wedge S^1$  is the swap.

**Proposition 1.4.** *There is a natural isomorphism in the stable homotopy category between  $\Sigma_f E$  and  $\Sigma E$ .*

To prove this, we use the notion of an  $S^2$ -spectrum [J, 9.4 p. 265]. An  $S^2$ -spectrum is a sequence of spaces  $X_n$  with  $n$  in  $\mathbb{Z}$  together with structure maps

$$S^2 \wedge X_n \rightarrow X_{n+1}.$$

Every spectrum  $E$  determines a  $S^2$ -spectrum  $R(E)$  by  $R(E)_n = E_{2n}$  with structure maps  $S^2 \wedge E_{2n} \rightarrow S^1 \wedge S^1 \wedge E_{2n} \xrightarrow{1 \wedge \epsilon_{2n}} S^1 \wedge E_{2n+1} \xrightarrow{\epsilon_{2n+1}} E_{2n}$ . Furthermore, every  $S^2$ -spectrum  $X$  determines an  $S^1$ -spectrum  $L(X)$  by  $L(X)_{2n} = X_n$ ,  $L(X)_{2n+1} = S^1 \wedge X_n$  with the structure map  $S^1 \wedge L(X)_{2n+1} \cong S^2 \wedge X_n \rightarrow L(X)_{2n+2} \cong X_{n+1}$  given by the structure map  $S^2 \wedge X_n \rightarrow X_{n+1}$  of  $X$ , and with the structure map  $S^1 \wedge L(X)_{2n} \cong S^1 \wedge X_n \rightarrow L(X)_{2n+1} \cong S^1 \wedge X_n$  given by the identity. There is a function

$$LR(E) \rightarrow E$$

which is the identity in evenly indexed levels, and the structure map  $S^1 \wedge E_{2n} \rightarrow E_{2n+1}$  in odd levels. The map  $LR(E) \rightarrow E$  is an isomorphism in the stable homotopy category because it induces an isomorphism on  $\pi_*$ .

*Proof.* By [J, 9.51], it suffices to show that  $R(\Sigma_f E)$  and  $R(\Sigma E)$  are isomorphic in the category of  $S^2$ -spectrum where the maps are homotopy classes of pmaps with the analogous definition of pmap and homotopy class.

Note that the spaces of  $R(\Sigma_f E)$  and  $R(\Sigma E)$  are the same, but their structure maps

$$S^2 \wedge S^1 \wedge E_{2n} \rightarrow S^1 \wedge S^1 \wedge S^1 \wedge E_{2n} \rightarrow S^1 \wedge E_{2n+2}$$

differ by the composition of two swaps on  $S^1 \wedge S^1 \wedge S^1$ . Since the composition of two swaps has degree  $(-1) \times (-1) = 1$ , there is a homotopy between these two structure maps. This allows us to define an  $S^2$ -spectrum  $H$  with  $n$ th space given by  $R(\Sigma E)_n \wedge [0, 1]_*$  and structure maps given as a homotopy between the structure maps of  $R(\Sigma_f E)$  and  $R(\Sigma E)$ . The inclusions  $\{0\}_* \hookrightarrow [0, 1]_*$  and  $\{1\}_* \hookrightarrow [0, 1]_*$  determine maps  $R(\Sigma_f E) \rightarrow H$  and  $R(\Sigma E) \rightarrow H$ . Both of these maps induce isomorphisms on homotopy groups by the contractibility of  $[0, 1]$ . Therefore, they are isomorphisms.  $\square$

**Remark 1.5.** *This phenomena is also discussed in [LMS, p. 42] (see also I.6.2 p. 35 for the definition of a  $w$ -map, and I.7.1 p. 40 for the definition of the shift desuspension) and in [AK].*

**Corollary 1.6.** *We may replace  $\Sigma_f$  by  $\Sigma$  in Proposition 1.3.*

**Corollary 1.7.** *Suspension*

$$\Sigma : [E, F]_* \rightarrow [\Sigma E, \Sigma F]_*$$

*induces a bijection.*

*Proof.* The isomorphisms of Proposition 1.3 induce a bijection

$$[\Sigma^{-1} \Sigma E, \Sigma^{-1} \Sigma F]_* \cong [E, F]_*$$

We have a map  $\Sigma^{-1} : [\Sigma E, \Sigma F]_* \rightarrow [\Sigma^{-1} \Sigma E, \Sigma^{-1} \Sigma F]_*$ , which is inverse to suspension.  $\square$

## 2 Smash products and internal function objects in the stable homotopy category

For pointed spaces  $X$  and  $Y$ , we have their smash product  $X \wedge Y = X \times Y / (* \times Y \cup X \times *)$  and the space of pointed maps  $F(X, Y) = \text{Map}_*(X, Y)$ , which is the internal function object. Note that there is a natural homeomorphism

$$F(X \wedge Y, Z) \cong F(X, F(Y, Z)). \quad (1)$$

**Fact 2.1.** *These statements all hold with  $X$ ,  $Y$ , and  $Z$  replaced by objects in the stable homotopy category.*

For example, if  $A$  and  $B$  are pointed spaces,  $\Sigma^\infty A \wedge \Sigma^\infty B = \Sigma^\infty(A \wedge B)$ .

We can use (1) and Brown representability to obtain our internal function object  $F(X, Y)$  for  $X$  and  $Y$  in the stable homotopy category. To do this: for a CW complex  $W$ , define  $X \wedge W$  by taking a spectrum representing  $X$  and then smashing each of the spectrum's spaces and structure maps with  $W$ . So,  $(X \wedge W)_n = X_n \wedge W$  and the structure map  $S^1 \wedge X_n \wedge W \rightarrow X_{n+1} \wedge W$  is  $\epsilon_n \wedge 1_W$ , where  $\epsilon_n : S^1 \wedge X_n \rightarrow X_{n+1}$  is the structure map of (the spectrum representing)  $X$ . One shows that  $W \mapsto [X \wedge W, Y]$  is a generalized reduced cohomology theory, and then Brown representability implies that it is represented by a spectrum, whose corresponding object in the stable homotopy category is  $F(X, Y)$ .

With the category of spectra we've been working with, there is no associative and commutative smash product, as was mentioned in Lecture 3. The construction on the stable homotopy category is given in [A, part III, 4], but it's messy. Because it's going to take work to set this up, we will first get some applications assuming it's existence. After we've solved some problems, we will go back to the construction, or at least there will be notes on it.

## References

- [AK] W. Abram, Igor Kriz *On shift desuspensions of Lewis-May-Steinberger spectra* New York Journal of Mathematics **18**, 2012, 55-58.
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- [J] R. Jardine, *Local Homotopy Theory* Springer, 2014.
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