Lecture 5: Homotopy groups and weak equivalences

1/14/14, 1/16/14

Let E be a spectrum with spaces E_n and structure maps $\epsilon_n : \Sigma E_n \to E_{n+1}$. The homotopy groups of E are defined

$$\pi_r(E) := \operatorname*{colim}_{n\to\infty} \pi_{r+n} E_n$$

where the maps implicit in this colimit take the class represented by $\gamma : S^{n+r} \to E_n$ to the class of $\epsilon_n \circ \Sigma\gamma$.

Note that by [A, III 2.8], proved in Lecture 4, $\pi_r E_n \cong [\mathbb{S}, E]_r$.

Example 1.1.
- *$\pi_n \mathbb{S}$ are the (stable) homotopy groups of spheres.*
- *$\pi_n MO \cong \Omega_n$ is the group of compact, oriented, smooth manifolds of dimension n up to the bordism equivalence relation by Thom's work.*

Our next goal is to show that homotopy groups control the isomorphisms in the stable homotopy category.

Theorem 1.2. *Let $E \to F$ be a function of spectra inducing an isomorphism on π_*. Then for any CW spectrum X, the map*

$$[X, E]_r \to [X, F]_r$$

is a bijection.

Corollary 1.3. *A pmap of CW spectra that induces an isomorphism on π_* is an isomorphism in the stable homotopy category.*

The analogous statement with "pmap" replaced by "function"' is an immediate consequence of Yoneda's lemma and Theorem 1.2. Thus, we know that "A function of CW spectra that induces an isomorphism on π_* is an isomorphism in the stable homotopy category." Here is a proof of the more general statement of Corollary 1.3.

Proof. Let E' be a cofinal subspectrm of E and let $f : E' \to F$ be a map of spectra inducing an isomorphism on π_*. By the theorem, we have $[X, E'] = [X, F]$. Taking $X = F$, the identity map on X determines an element of $[X, F]$. Thus there is $g : F' \to E'$ where F' is some cofinal sub spectrum of F, and such that fg is the homotopic to the identity as a pmap $F \to F$.

Taking $X = E'$, we see that $[E', E'] \to [E', F]$ is a bijection. Note that $gf : E' \to E'$ maps to fgf in $[E', F]$. Since $fg = 1$, we have $fgf = f$. By the injectivity of this bijection, we have that $gf = 1$. Thus f and g are inverse isomorphisms. □

To prove Theorem 1.2, we will use Adam's technique of induction over *stable cells*. Let C_n denote the set of cells of the CW complex E_n. ϵ_n induces a map $C_n \to C_{n+1}$. The stable cells are $C = \mathrm{colim}_{n\to\infty} C_n$. Note that E' is cofinal in E if and only if the map of stable cells is a bijection. In order to induct, Adam's uses [A, III Lemma 3.1]:

Lemma 1.4. *If $E \subset F$ is an inclusion of CW spectra such that E is not cofinal. Then there exists a sub-CW-spectrum J of F such that $E \subset J \subset F$ and such that J contains exactly one more stable cell than E.*

Proof. Since E is not cofinal, there is a stable cell of F not in E. This cell has a representative c in some F_n. c is contained in a finite subcomplex of F_n. Thus there are finite complexes of F_n with stable cells not contained in E. Choose such a finite complex K with the fewest number of cells. $K \subset E_n$ for some n. By minimality, K has a unique top dimensional cell which does not represent a stable cell in E, and the lower dimensional cells represent stable cells contained in E. Let $K = L \cup c'$ denote the decomposition of K into it's top dimensional cell c' and its lower cells L. After some (finite) number of suspensions, we have $\Sigma^m K = \Sigma^m L \cup \Sigma^m c'$ with $\Sigma^m L \subset E_{n+m}$. Let $J_i = E_i \cup \Sigma^{i-n} c'$ for $i \geq n + m$ and $J_i = E_i$ for $i < n + m$. □

Lemma 1.5. *Let $f : E \to F$ be a function of spectra inducing an isomorphism on π_*. Let $i : A \hookrightarrow X$ be an inclusion of CW spectra. Let $g : X \to F$ be a pmap such that $gi = f\phi$ for a pmap $\phi : A \to E$. Then there is a pmap $H : \mathrm{Cyl}(X) \to F$ such that H is a homotopy between g and $f\theta$, where $\theta : X \to E$ is a pmap which restricts to ϕ. We may furthermore have the restriction of H to* $\mathrm{Cyl}(A)$ *is the constant homotopy on $f\phi$.*

By the "constant homotopy on $f\phi$," we mean that for any $t \in [0, 1]$, there is a map $t_+ \to [0, 1]_+$. This map induces a map $A \to \mathrm{Cyl}(A)$. The constant homotopy means that when we pull back along any of these maps we get $f\phi$. Alternatively, $\mathrm{Cyl}(A)_n = [0, 1]_+ \wedge A_n = (A_n \times [0, 1])/(* \times [0, 1])$. A pointed map on A induces a map on A_n, whence on $A_n \times [0, 1]$, and since the original map was pointed, we obtain a map on $\mathrm{Cyl}(A)_n$, whence a map on $\mathrm{Cyl}(A)$. For a pmap, this gives a map on $\mathrm{Cyl}(A')$ for $A' \subset A$ cofinal. Since $\mathrm{Cyl}(A')$ is cofinal in $\mathrm{Cyl}(A)$, this map on $\mathrm{Cyl}(A')$ is a pmap on $\mathrm{Cyl}(A)$. This pmap is the constant homotopy.

Proof. By replacing A and X by cofinal sub spectra, we may assume that ϕ and g are represented by functions on A and X respectively.

Consider the set of triples (B, h, η) where

- B is a sub-CW-spectrum of X containing A. Consider B to be a subspectrum of $\mathrm{Cyl}(B)$ under the inclusion corresponding to $t = 0$.
- a map $h : \mathrm{Cyl}(B) \to F$ such that the restriction of h to $B \cup \mathrm{Cyl}(A)$ is given by g and the constant homotopy on $f\phi$.
- A function $\eta : B_1 \to E$ such that $f\eta = h|_{B_1}$, where B_1 denotes the subspectrum of $\mathrm{Cyl}(B)$ which is the copy of B under the inclusion corresponding to $t = 1$.

Note that $(A, \text{constant homotopy}, \phi)$ is one such triple, so this set is nonempty. There is a partial ordering on the set of triples, defined $(B, h, \eta) \leq (B', h', \eta')$ if $B \subseteq B'$ and the restrictions of h' and η' are h and η respectively. Note that any chain has an upper bound given by taking the union of the sub-CW-spectra and the functions whose restrictions to the subspectra comprising the union are as required to be an upper bound for the chain. By Zorn's lemma, we therefore have a maximal element (B, h, η). To prove the lemma, if suffices to show that B is cofinal.

Suppose the contrary. Then by Lemma 1.4, there is a subspectrum J containing one more stable cell than B. Choose n such that J_n has a representative of the stable cell not in B. Call this representative cell c. c is the single top cell of a finite complex $K = L \cup c$ in J_n. After a finite number m of suspensions we will have that $J_{n+m} \supset \Sigma^m L \cup \Sigma^m c$ is such that $\Sigma^m L \subset B_{n+m}$. By replacing J by its subsepectrum given as the union of E and suspensions of $\Sigma^m L \cup \Sigma^m c$, we may assume that the only cells of J not in E are suspensions of a single cell $\Sigma^m c$.

To contradict maximality of B, it therefore suffices to extend η_k and h_k to $B_k \cup \Sigma^{k-n} c$ and $\mathrm{Cyl}(B_k \cup \Sigma^{k-n} c)$ respectively for some large enough k. Note that $h| \cup g| : (\partial c) \wedge [0,1]_+ \cup c \times 0$ defines an extension of h to $(\partial c) \wedge [0,1]_+ \cup c \times 0$. Note that $(\partial c) \wedge [0,1]_+ \cup c \times 0$ is equivalent to a ball of the same dimension as c. Thus the image of $\partial c \times 1$ under h is null in $\pi_* F$. Since $\pi_* E \to \pi_* F$ is an isomorphism, this implies that the image of $\partial c \times 1$ under g is null in $\pi_* E$.

This implies that after sufficiently many suspensions, we may extend η. Then $h|\cup g|\cup f\eta : (\partial\Sigma^k c)\wedge[0,1]_+\cup\Sigma^k c\times 0\cup\Sigma^k c\times 1$ determines an element γ of $\pi_{*+1}F$. Since $\pi_{*+1}E \to \pi_{*+1}F$ is an isomorphism, we may lift $-\gamma$ to E after sufficiently many suspensions. Adding this lift to η, we obtain a new η and k such that the element of $\pi_{*+1}F$ determined by $h|\cup g|\cup f\eta : (\partial\Sigma^k c)\wedge[0,1]_+\cup\Sigma^k c\times 0\cup\Sigma^k c\times 1$ is null-homotopic. A choice of null-homotopy gives the desired extension of h_k for some sufficiently large k. ☐

Exercise 1.6. *Use the argument just given to prove the homotopy extension property: Let X, A be a pair of CW-spectra. Suppose given a map $f : X \to Y$ and a homotopy $h : \mathrm{Cyl}(A) \to Y$ such that $f|_A = h|_{A\times 0}$. Then the homotopy can be extended over* $\mathrm{Cyl}(X)$.

Proof. (of theorem 1.2) For any pmap $g : X \to F$, the image of the base-point is in E. Thus applying Lemma 1.5 with $(X, A) = (X, * = \Sigma^\infty *)$ and the unique map $\phi : * \to E$, we see there is a pmap $\theta : X \to E$ such that g is homotopic to $f\theta$. This shows surjectivity.

Injectivity: suppose $\theta_i : X \to E$ for $i = 1, 2$ are pmaps which map to the same element of $[X, F]$. Thus there exists a pmap $H : \mathrm{Cyl}(X) \to F$ giving a homotopy between $f\theta_1$ and $g\theta_2$. We have the inclusions $X \to \mathrm{Cyl}(X)$ for any $t \in [0, 1]$ from above. Taking $t = 0, 1$, we have $X \vee X \hookrightarrow \mathrm{Cyl}(X)$. Apply Lemma 1.5 with $(X, A) = (\mathrm{Cyl}(X), X \vee X)$ and $\phi : X \vee X \to E$ given $\phi = \theta_1 \vee \theta_2$. We obtain a pmap H', homotopic to H, with $H' : \mathrm{Cyl}(X) \to E$ and such that the restriction of H' to $X \vee X$ is ϕ. thus θ_1 is homotopic to θ_2. ☐

References

[A] J.F. Adams, *Stable Homotopy and Generalized Homology* Chicago Lectures in Mathematics, The University of Chicago Press, 1974.