Lecture 4: Generalized cohomology theories

1/12/14

We've now defined spectra and the stable homotopy category. They arise naturally when considering cohomology.

Proposition 1.1. *For X a finite CW-complex, there is a natural isomorphism* $[\Sigma^\infty X, H\mathbb{Z}]_{-r} \cong \mathrm{H}^r(X, \mathbb{Z})$.

The assumption that X is a finite CW-complex is not necessary, but here is a proof in this case. We use the following Lemma.

Lemma 1.2. *([A, III Prop 2.8]) Let F be any spectrum. For X a finite CW-complex there is a natural identification* $[\Sigma^\infty X, F]_r = \mathrm{colim}_{n\to\infty}[\Sigma^{n+r}X, F_n]$

On the right hand side the colimit is taken over maps $[\Sigma^{n+r}X, F_n] \to [\Sigma^{n+r+1}X, F_{n+1}]$ which are the composition of the suspension $[\Sigma^{n+r}X, F_n] \to [\Sigma^{n+r+1}X, \Sigma F_n]$ with the map $[\Sigma^{n+r+1}X, \Sigma F_n] \to [\Sigma^{n+r+1}X, F_{n+1}]$ induced by the structure map of F $\Sigma F_n \to F_{n+1}$.

Proof. For a map $f_{n+r} : \Sigma^{n+r}X \to F_n$, there is a pmap of degree r of spectra $\Sigma^\infty X \to F$ defined on the cofinal subspectrum whose mth space is $\Sigma^m X$ for $m \geq n+r$ and $*$ for $m < n+r$. This pmap is given by $\Sigma^{m-n-r} f_{n+r}$ for $m \geq n+r$ and is the unique map from $*$ for $m < n+r$. Moreover, if $f_{n+r}, f'_{n+r} : \Sigma^{n+r}X \to F_n$ are homotopic, we may likewise construct a pmap $\mathrm{Cyl}(\Sigma^\infty X) \to F$ of degree r. Thus there is a well defined map $[\Sigma^{n+r}X, F_n] \to [\Sigma^\infty X, F]_r$.

Note that, by construction, the image of f_{n+r} under

$$[\Sigma^{n+r}X, F_n] \to [\Sigma^{n+r+1}X, F_{n+1}] \to [\Sigma^\infty X, F]_r$$

equals the image of f_{n+r} under

$$[\Sigma^{n+r}X, F_n] \to [\Sigma^\infty X, F]_r$$

when restricted to the cofinal subspectrum of $\Sigma^\infty X$ whose mth space is $\Sigma^m X$ for $m \geq n+r+1$ and $*$ for $m < n+r-1$. In particular, these images are equal as pmaps of degree r. Thus we have a well defined function

$$\theta : \operatorname{colim}_{n\to\infty}[\Sigma^{n+r}X, F_n] \to [\Sigma^\infty X, F]_r.$$

θ is surjective as follows. Let K be a cofinal subspectrum of $\Sigma^\infty X$ and let $g : K \to F$ be a function of spectra of degree r. Since K is cofinal and X is a finite complex, there exists an m such that $\Sigma^m X \subset K_m$. Thus g is in the image of $[\Sigma^m X, F_{m-r}]$, showing surjectivity.

θ is injective by applying the surjectivity of θ for the pmaps of degree r from $\mathrm{Cyl}(\Sigma^\infty X) = \Sigma^\infty \mathrm{Cyl}(X)$ to F.

□

Proof. (of Proposition 1.1) By Lemma 1.2, $[\Sigma^\infty X, H\mathbb{Z}]_{-r} = \operatorname{colim}_{n\to\infty}[\Sigma^{n-r}X, K(\mathbb{Z}, n)]$. The adjunction between Σ and Ω gives an equivalence of topological spaces $\mathrm{Map}(\Sigma\Sigma^{n-r}X, K(\mathbb{Z}, n+1)) \cong \mathrm{Map}(\Sigma^{n-r}X, \Omega K(\mathbb{Z}, n+1))$. Applying π_0, we have a bijection $[\Sigma^{n-r+1}X, K(\mathbb{Z}, n{+}1)] = [\Sigma^{n-r}X, \Omega K(\mathbb{Z}, n{+}1)]$. Since $\Omega K(\mathbb{Z}, n+1) \cong K(\mathbb{Z}, n)$, we have a bijection $[\Sigma^{n-r+1}X, H\mathbb{Z}_{n+1}] = [\Sigma^{n-r}, H\mathbb{Z}_n]$. Unwinding definitions, we see that this bijection is compatible with the transition maps of the colimit $\operatorname{colim}_{n\to\infty}[\Sigma^{n-r}X, H\mathbb{Z}_n]$. Thus $\operatorname{colim}_{n\to\infty}[\Sigma^{n-r}X, H\mathbb{Z}_n] = [X, H\mathbb{Z}_r] \cong \mathrm{H}^r(X, \mathbb{Z})$ where the last isomorphism is the representability of cohomology discussed before. □

Let **CW** denote the category whose objects are CW-complexes with a 0-cell chosen as a base-point and whose maps are basepoint preserving maps. Let **Ab** denote the category of abelian groups.

Definition 1.3. *([H, 4.E]) A* reduced cohomology theory *on* **CW** *is a sequence of functors $e^n : \mathbf{CW} \to \mathbf{Ab}$ together with natural isomorphisms $e^n(X) \cong e^{n+1}\Sigma X$ for all $X \in \mathbf{CW}$ such that the following axioms hold*

1. *If $f, g : X \to Y$ are homotopic (preserving base points) then they induce the same maps $e^n Y \to e^n X$.*
2. *For each inclusion $A \hookrightarrow X$ in* **CW***, the sequence $e^n X/A \to e^n X \to e^n A$ is exact.*
3. *For a wedge sum $X = \vee_\alpha X_\alpha$ with inclusions $\iota_\alpha : X_\alpha \to X$, the product map $\prod_\alpha \iota_\alpha : e^n(X) \to \prod_\alpha e^n(X_\alpha)$.*

The representability theorem of Brown [B] says that the functors e^n in every reduced cohomology theory are represented in the homotopy category of spaces, and that, furthermore, the representing spaces form an Ω-spectrum.

Theorem 1.4. *(Brown, [H, Theorem 4E.1]) Let e^n be a reduced cohomomogy theory on* **CW**. *Then there exists based spaces E_n and natural isomorphisms $e^n X = [X, E_n]$ so that the spaces E_n form an Ω-spectrum.*

We won't give the proof in class, but we can see that the spaces E_n form an Ω-spectrum with the argument we just used to prove Proposition 1.1. Namely, $[X, E_n] \cong e^n(X) \cong e^{n+1}(\Sigma X) \cong [\Sigma X, E_{n+1}] \cong [X, \Omega E_{n+1}]$ is a natural isomorphism. It follows by Yoneda's lemma that there is an equivalence $E_n \cong \Omega E_{n+1}$. (To see this, take $E = \Omega E_{n+1}$ on the right hand side. The identity map on ΩE_{n+1} gives a map $\Omega E_{n+1} \to E_n$ from the left hand side. Starting from the left, we similarly obtain a map $E_n \to \Omega E_{n+1}$. Then check the compositions are the identity. Here is this argument as an exercise.

Exercise 1.5. *Prove Yonda's lemma: let $\mathbf{C}$ be a category, and let $\mathrm{Fun}(\mathbf{C}^{\mathrm{op}}, \mathbf{Set})$ denote the category of functors from $\mathbf{C}^{\mathrm{op}}$ to* **Set**. *Define the functor $h : \mathbf{C} \to \mathrm{Fun}(\mathbf{C}^{\mathrm{op}}, \mathbf{Set})$ by $h(C) = \mathrm{Map}(-, C)$. Show that h is fully-faithful, i.e., $\mathrm{Map}(C, D) = \mathrm{Map}(h(C), h(D))$.*

)

Just as in the proof of 1.1, it follows from Proposition 1.2 that for a finite complex X, we have $[X, E]_{-r} = e^r(X)$, and as above the hypothesis that X be finite is not necessary. This motivates the following definition.

Definition 1.6. *For a spectrum E, define the generalized E-cohomolgoy of degree r of a spectrum X to be $E^r X = [X, E]_{-r}$.*

Example 1.7.
1. *When $E = \mathbb{S}$, the generalized E-chomology is* stable cohomotopy.
2. *Generalized MU-cohomology is called complex cobordism, Generalized MO-chomology is cobordism.*

References

[A] J.F. Adams, *Stable Homotopy and Generalized Homology* Chicago Lectures in Mathematics, The University of Chicago Press, 1974.

[B] Edgar Brown *Cohomology Theoreis*, The Annals of Mathematics, 2nd Ser. Vol. 75, No. 3. (May 1962) pp 467-484.

[H] Allen Hatcher, *Algebraic Topology.*