Lecture 3: the stable homotopy category

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The *stable homotopy category* is the category of spectra where the morphisms, instead of being maps of spectra, are something like homotopy classes of maps of spectra.

There are many constructions of the stable homotopy category. Here is the one in Adams [A, Part III. 2]. This construction starts from a category of spectra, and then changes the maps (which he calls "functions") to something like homotopy classes of maps (which he calls "morphisms"). The reason that there are other constructions is that there is a problem with this one. It is important for the stable homotopy category to have an associative and commutative smash product denoted $\wedge$ (or sometimes $\otimes$ because of similarities with the tensor product). For example, this smash product appears in the definition of the generalized homology theory associated to a spectrum, as well as in the definition of the cup product, and in duality theorems. The problem with the category of spectra given here is that it does not have an associative and commutative smash product, and then the resulting construction of an associative and commutative smash product on the stable homotopy category is problematic (but it exists).

1 Homotopy category of spectra

Definition 1.1. *E is a CW-spectrum if*

- *for each n, E_n is a CW-complex with base point.*
- *each map $\epsilon_n : \Sigma E_n \to E_{n+1}$ is an isomorphism from ΣE_n to a sub complex of E_{n+1}.*

Remark 1.2. *When we have a definition of weak equivalence, we will see that any spectrum is weakly equivalent to a CW-spectrm. See [A, Part III, Exercise after 3.12]. So it is okay to restrict attention to CW-spectra.*

A *subspectrum* A of a CW-spectrum E is a spectrum with $A_n \subset E_n$ a subcomplex.

A is said to be *cofinal* in E if for each n and each finite subcomplex $K \subset E_n$ there is an m such that $\Sigma^m K$ maps into A_{m+n} under the canonical map

$$\Sigma^m E_n \stackrel{\Sigma^{m-1}\epsilon_n}{\to} \Sigma^{m-1} E_{n+1} \stackrel{\Sigma^{m-2}\epsilon_{n+1}}{\to} \ldots \stackrel{\epsilon_{n+m-1}}{\to} E_{m+n}$$

If E and E' are two cofinal subspectra of E, and $f' : E' \to F$ and $f'' : E'' \to F$ are two maps, say that f' and f'' are *equivalent* if there is a cofinal sub spectrum E'' contained in E' and E'' such that the restrictions of f' and f'' to E''' are equal. Since the intersection of two cofinal subspectra is a cofinal subspectra, this defines an equivalence relation.

Adams calls the set of equivalence classes of maps from all cofinal subspectra of E to F the "maps" from E to F. Since we've been using the word "map" as a synonym for function, let's use another word.

Definition 1.3. *An equivalence class of a map from a cofinal subspectrum of E to F will be called a* pmap *from E to F.*

Example 1.4. *Let $\mathbb{S}$ denote the* sphere spectrum *meaning the suspension spectrum of S^0. Let $K \subset \mathbb{S}$ be the subspectrum with $K_n = *$ for $n \leq 2$ and $K_n = \mathbb{S}^n$ for $n \geq 3$. Let $\eta : S^3 \to S^2$ be the Hopf map. $\Sigma^n \eta$ defines a map $K_{n+3} \to \mathbb{S}_{n+2}$ for $n \geq 0$. For $n < 0$, $K_{n+3} = *$ is a point and there is a unique map $K_{n+3} \to \mathbb{S}_{n+2}$. These maps $K_{n+3} \to \mathbb{S}_{n+2}$ are the data of a pmap of degree 1 from $\mathbb{S}$ to $\mathbb{S}$. Equivalently, this data gives a pmap $\mathbb{S} \to \Sigma^{-1}\mathbb{S}$, although technically we haven't defined $\Sigma^{-1}\mathbb{S}$.*

Proposition 1.5. *Composition of maps ("functions") of spectra determines a well-defined composition of pmaps.*

Proof. Let F' be a cofinal subspectrum of a spectrum F and $E \to F$ a function of spectra (or a function of degree $r \neq 0$). We claim that there is a cofinal subspectrum of E which maps into F'. Note that there is a largest CW subcomplex E'_n of E_n which is mapped to F', i.e. a cell is included precisely if it and all the lower dimensional cells required by attaching maps are mapped into F'. By the definition of a function of spectra, and the fact that F' is a subspectrum, we must have $\epsilon_n(\Sigma E'_n) \subset E'_{n+1}$, where $\epsilon_n : \Sigma E_n \to E_{n+1}$ is the structure map in E. Thus there is a largest CW subspectrum E' of E mapping to F'. We claim that this subspectrum is cofinal. Let K be a finite sub complex of E_n. The image of K is compact, and therefore is contained in a finite sub complex K_F of F_n. See for example [H, Prop A.1 p 520]. Since F' is cofinal, there is an m as above for K_F. Since $\Sigma^m K$ maps into $\Sigma^m K_F \subset F'$, we have that $\Sigma^m K$ maps into F', from which it follows that $\Sigma^m K$ is in E', showing E' is cofinal.

Given two pmaps $f : E \to F$ and $g : F \to H$, we have cofinal sub spectra F' and E' of F and E on which g and f are defined respectively. By the above, there is a cofinal subspectrum E'' of E' mapping into F' under f. It is straightforward to check that a cofinal subspectrum of a cofinal subspectrum is cofinal. Thus E'' is a cofinal subspectrm of E. We may define $g \circ f$ on E''. $\square$

Now we wish to define the homotopy class of a pmap. For a space X, let $X_+ = X \coprod *$. Let $\mathrm{Cyl}(E)$ denote the spectrum whose nth space is $[0,1]_+ \wedge E_n$ and with structure maps given by the flip $S^1 \wedge [0,1]_+ \wedge E_n \to [0,1]_+ \wedge S^1 \wedge E_n$ composed with $[0,1]_+ \wedge \epsilon_n$. The maps $\{0\}_+ \to [0,1]_+$ and $\{1\}_+ \to [0,1]_+$ allow us to define two maps $E \to \mathrm{Cyl}(E)$. Two pmaps $f, g : E \to F$ are *homotopic* if there is a pmap $H : \mathrm{Cyl}(E) \to F$ such that H precomposed with our maps $E \to \mathrm{Cyl}(E)$ produces f and g. The standard proof that homotopy is an equivalence relation applies.

Definition 1.6. *A morphism $E \to F$ in the stable homotopy category from a CW-spectrum E to F is a homotopy class of pmaps. A morphism of degree r is a homotopy class of a degree r pmap. Let $[E, F]$ denote the morphisms from E to F in the stable homotopy category, and let $[E, F]_r$ denote the morphisms of degree r.*

Next: Generalized (co)homology. In particular, for a CW-complex X, we will see

Fact 1.7. $[\Sigma^\infty X, H\mathbb{Z}]_{-r} \cong \mathrm{H}^r(X, \mathbb{Z})$.

References

[A] J.F. Adams, *Stable Homotopy and Generalized Homology* Chicago Lectures in Mathematics, The University of Chicago Press, 1974.

[H] Allen Hatcher, *Algebraic Topology.*