Lecture 2: Spectra

1/7/14

We are going to study stable phenomena in homotopy theory, so we build a category of *Spectra* where homotopy theory of spaces makes sense and where suspension is forced to be an equivalence. There are a number of equivalent points of view on spectra. Here is the one in Adams [A, Part III. 2].

1 Category of Spectra

Definition 1.1. *A* spectrum E *is a sequence of spaces* E_n *with a basepoint together with maps*

$$\epsilon_n : S^1 \wedge E_n = \Sigma E_n \to E_{n+1}.$$

The index n may vary over $\mathbb{Z}$ or $\mathbb{Z}_{\geq 0}$. Let's have n vary over $\mathbb{Z}$ to be definite.

For a based space Y, let $\Omega Y = \mathrm{Map}_*(S^1, Y)$ denote the topological space of based maps $S^1 \to Y$. Note that there is a natural equivalence

$$\mathrm{Map}_*(\Sigma X, Y) \cong \mathrm{Map}(X, \Omega Y)$$

for any based spaces X and Y. Thus the maps ϵ_n are equivalent to the data of maps

$$\epsilon_n' : E_n \to \Omega E_{n+1}.$$

If E_n is connected, then the image of ϵ_n' will automatically lie inside the connected component of the base point in ΩE_{n+1}. Let $\Omega_0 E_{n+1}$ denote the connected component of the base point. If all our spaces are connected, we could then consider ϵ_n' as a map to $\Omega_0 E_{n+1}$.

Definition 1.2. *A map ("function of degree* 0*" in [A])* $E \to F$ *of spectra is a* *sequence of maps $f_n : E_n \to F_n$ such that*

$$\begin{array}{ccc} \Sigma E_n & \xrightarrow{\epsilon_n} & E_{n+1} \\ \downarrow{\scriptstyle f_n} & & \downarrow{\scriptstyle f_{n+1}} \\ \Sigma F_n & \xrightarrow{\epsilon_n} & F_{n+1} \end{array}$$

commutes.

This diagram is required to commute strictly, as opposed to commuting up to homotopy.

We do not yet have a notion of homotopy classes of maps or a good way to do homotopy theory on this category. We will start to do that on Friday.

It's useful to also have a notion of a *function of degree r $E \to F$*. This is a sequence of maps $f_n : E_n \to F_{n-r}$ such that the analogous diagram commutes.

1.1 Examples

1.2. *Suspension spectra.* Given a space X, the suspension spectrum $\Sigma^\infty X$ of X is the spectrum with $(\Sigma^\infty X)_n = \begin{cases} \Sigma^n X \cong S^n \wedge X & \text{if } n \geq 0 \\ * & \text{if } n < 0. \end{cases}$.

1.3. *Eilenberg-MacLane spectra.* An Eilenberg-MacLane space of type (π, n) is a space $K(\pi, n)$ with a base point such that

$$\pi_*(K(\pi, n)) = \begin{cases} \pi & \text{if } * = n \\ 0 & \text{otherwise.} \end{cases}$$

For any space X, the sequence

$$0 \to \mathrm{Ext}(\mathrm{H}_{n-1}(X), \pi) \to \mathrm{H}^n(X, \pi) \to \mathrm{Hom}(\mathrm{H}_n(X), \pi) \to 0$$

is exact by the universal coefficient theorem for cohomology. Thus if X is $(n-1)$-connected, $\mathrm{H}^n(X, \pi) \cong \mathrm{Hom}(\mathrm{H}_n(X), \pi)$. When $\pi_n(X) = \pi$, the inverse of the Hurewicz homomorphism is an element of the right hand side, whence we have an element of $\mathrm{H}^n(X, \pi)$. Call this element ι.

Theorem 1.3. *(Representability of cohomology) For any Y, the map $[Y, K(\pi, n)] \to \mathrm{H}^n(Y, \pi)$ defined by $[f] \mapsto f^*\iota$ is a bijection.*

This is an nice illustration of obstruction theory. See, for example, [MT, Theorem 1, chapter 1] or [H, Theorem 4.57].

Corollary 1.4. *There is a canonical homotopy equivalence between any two* $K(\pi, n)$ *spaces.*

Since $\Omega K(\pi, n+1)$ is a $K(\pi, n)$-space, we therefore have map $\epsilon'_n : K(\pi, n) \to \Omega K(\pi, n+1)$.

Let $H\pi$ be the spectrum whose nth space is $K(\pi, n)$ together with the maps ϵ'_n. This is the Eilenberg-MacLane spectrum for the group π.

Spectra where the maps ϵ'_n are weak equivalences are called *omega spectra*, so in particular, $H\pi$ is an omega spectrum.

1.4. *Thom spectra: MO, MU, MSO etc..*

For a vector bundle $E \to X$, we can choose a continuously varying norm on the fibers. Then we have the associated disk bundle $D(E) = \{e \in E : |e| \leq 1\}$ and the associated sphere bundle $S(E) = \{e \in E : |e| = 1\}$

Definition 1.5. *Let* $\mathrm{Th}(E) = D(E)/S(E)$ *be the* Thom space *of* E.

If X is a compact CW-complex, then $\mathrm{Th}(E)$ is homotopy equivalent to E^+ the one point compactification of E. (Proof: $D(E)$ is compact because it is a fiber bundle with compact base and fiber, thus $\mathrm{Th}(E)$ is compact. E is homotopy equivalent to $D(E) - S(E)$ using a homotopy equivalence of $\mathbb{R}^n$ with the open ball. Note that $D(E) - S(E)$ is a subspace of $\mathrm{Th}(E)$ whose complement is a point. To see that the topology is right, one could proceed as follows: E^+ is the terminal compactfication, so there is a continuous map $\mathrm{Th}(E) \to E^+$. We have just seen that this is a bijection. Since E^+ is Hausdorff and $\mathrm{Th}(E)$ is compact, $\mathrm{Th}(E) \to E^+$ is therefore a homeomorphism.)

Let $\underline{1}$ be the trivial real vector bundle of rank 1 on X. By choosing a supmetric, we can have $D(\underline{1} \oplus E) \cong [0,1] \times D(E)$, and $S(\underline{1} \oplus E) \cong \partial D(\underline{1} \oplus E) = \{0,1\} \times D(E) \cup [0,1] \times S(E)$. Thus $\mathrm{Th}(\underline{1} \oplus E) \cong$

$$\frac{[0,1] \times D(E)}{\{0,1\} \times D(E) \cup [0,1] \times S(E)} \cong \frac{S^1 \times D(E)}{* \times D(E) \cup S^1 \times S(E)} \cong \frac{S^1 \times \mathrm{Th}(E)}{* \times \mathrm{Th}(E) \cup S^1 \times *} = S^1 \wedge \mathrm{Th}(E).$$

Exercise 1.6. *If* $p : E \to X$ *and* $p' : E' \to Y$ *are two sphere bundles, we can define the fiber wise join*

$$E \hat{*} E' = \frac{E \times [0,1] \times E'}{\sim},$$

where $(x,1,y) \sim (x,1,y')$ if $p'y = p'y$ and $(x,0,y) \sim (x',0,y)$ if $px = px'$. You can check that $E * E'$ is a sphere bundle over $X \times Y$. The Thom space of a sphere bundle $E \to B$ is its mapping cone. Show that

1. $\mathrm{Th}(S(V)) \cong \mathrm{Th}\, V$ *for a vector bundle* V
2. $\mathrm{Th}(X \times S^{n-1}) \cong \Sigma^n X_+$*, where* $X_+ = X \coprod *$
3. $\mathrm{Th}(E \hat{*} E') \cong \mathrm{Th}(E) \wedge \mathrm{Th}(E')$.

A model for $BO(n)$ is the Grassmannian of n-planes in $\mathbb{R}^\infty$, meaning $BO(n) = EO(n)/O(n)$ where $EO(n)$ is the colimit $m \to \infty$ of the space of orthogonal n-tuples in $\mathbb{R}^m$. Let $\zeta_n \to BO(n)$ be the vector bundle whose fiber over a point of $BO(n)$ is the vector space spanned by the orthogonal n-tuples. ζ_n is called the *tautological bundle* over the classifying space of $BO(n)$, or the *universal* real vector bundle of rank n. Any rank n real vector bundle over a space Y is the pull-back of ζ_n under some map $Y \to BO(n)$. See [H2, Theorem 1.16]. Thus the $n+1$ bundle $\underline{1} \oplus \zeta_n$ maps to ζ_{n+1}. Applying Th, we have $\epsilon_n : \Sigma\,\mathrm{Th}(\zeta_n) = \mathrm{Th}(\underline{1} \oplus \zeta_n) \to \mathrm{Th}(\zeta_{n+1})$.

Definition 1.7. *MO is the spectrum whose nth space is $\mathrm{Th}(\zeta_n)$ and structure maps ϵ_n*

Definition 1.8. *MU is the spectrum whose $2n$th space is the Thom space of the tautological bundle over $BU(n)$, and whose $(2n+1)$st space is the suspension of the $(2n)$th space.*

Alternatively, we could construct MU as follows. Once Σ is an equivalence, it makes sense to define a spectrum $MU(n) = \Sigma^{-2n}\,\mathrm{Th}(\zeta_n)$ where ζ_n now denotes the tautological bundle over $BU(n)$. There is a canonical map

$$MU(n-1) \cong \Sigma^{2-2n}\,\mathrm{Th}(\zeta_{n-1}) \cong \Sigma^{-2n}\,\mathrm{Th}(\zeta_{n-1} \oplus \underline{1}) \to \Sigma^{-2n}\,\mathrm{Th}(\zeta_n) \cong MU(n).$$

Then $MU = \mathrm{colim}_{n\to\infty} MU(n)$.

References

[A] J.F. Adams, *Stable Homotopy and Generalized Homology* Chicago Lectures in Mathematics, The University of Chicago Press, 1974.

[H] Allen Hatcher, *Algebraic Topology.*

[H2] Allen Hatcher, *Vector bundles and K-theory.*

[MT] Robert Mosher and Martin Tangora, *Cohomology Operations and Applications in Homotopy Theory*, Dover, 1968, 2008.