Lecture 23: The Hopf invariant one problem

4/20/15-4/27/15

We consider the example of the Adams spectral sequence for $H\mathbb{F}_2$ and the stable homotopy groups of the sphere spectrum $\pi_*(S)$

$$E^2_{s,t} = \mathrm{Ext}^s_{\mathcal{A}}(H\mathbb{F}_2^* S, H\mathbb{F}_2^* \Sigma^t S) \Rightarrow \pi_{t-s} S$$

We will see that the potential elements of Hopf invariant one lie on $s = 1$, and discuss the Hopf invariant one problem.

We need some information about $\mathcal{A}$.

1 The Steenrod Algebra

Recall that $\mathcal{A} = H\mathbb{F}_2^* H\mathbb{F}_2$ is the Steenrod algebra. It can be computed in terms of generators and relations in the following manner. By Brown representability, elements of $\mathcal{A}$ are natural transformations from ordinary cohomology with $\mathbb{F}_2$-coefficients to itself that increase the grading by $*$. One can construct certain elements Sq^i of degree i for $i \geq 0$ with $\mathrm{Sq}^0 = 1$ of $\mathcal{A}$ [MT, Ch 2,3] and show the Adem relation

$$0 = R(a,b) = \mathrm{Sq}^a \mathrm{Sq}^b + \Sigma_c \binom{b-c-1}{a-2c} \mathrm{Sq}^{a+b-c} \mathrm{Sq}^c$$

for $0 < a < 2b$. Using representability of cohomology, one can see that the Sq^i generate all of $\mathcal{A}$ and the $R(a,b)$ are all the relations by computing $H\mathbb{F}_2^* K(\mathbb{Z}/2, n)$ for all n. The answer is then (see [Ma, Ch 15] or [MT, Ch 2,3,6] or ...):

Theorem 1.1. *There is an isomorphism*

$$\mathcal{A} \cong T(\mathrm{Sq}^i : i \geq 0)/\langle 1 + \mathrm{Sq}^0, R(a,b) : 0 < a < 2b\rangle,$$

where $T(\mathrm{Sq}^i : i \geq 0)$ denotes the tensor algebra over $\mathbb{Z}/2$ with generators Sq^i with $i \geq 0$, and where $\langle 1 + \mathrm{Sq}^0, R(a,b) : 0 < a < 2b\rangle$ denotes the two sided ideal.

We will also use the following facts about the Sq^i, which can be found in [MT, Ch 3] and [H, 4.L]:

1. $\mathrm{Sq} : H^*(X, \mathbb{F}_2) \to H^*(X, \mathbb{F}_2)$ defined by $\mathrm{Sq}(x) = \Sigma_i \mathrm{Sq}^i x$ is a ring homomorphism for all CW-complexes X.
2. For $x \in H^i(X, \mathbb{F}_2)$, we have $\mathrm{Sq}^i x = x^2$ and $\mathrm{Sq}^n x = 0$ for all $n > i$.

2 Computing Adams spectral sequence $\mathrm{Ext}^s_{\mathcal{A}}(H\mathbb{F}_2^* S, H\mathbb{F}_2^* \Sigma^t S) \Rightarrow \pi_{t-s} S \otimes \mathbb{Z}_2$

We commute $E^2_{s,t}$ for $s = 0, 1$. Note that $H\mathbb{F}_2^* S$ is $\mathbb{F}_2$ in dimension 0 and 0 otherwise. Similarly $H\mathbb{F}_2^* \Sigma^t S$ is $\mathbb{F}_2$ in dimension t and 0 otherwise. Let $\mathbb{F}_2[t]$ denote this module, i.e., $\mathbb{F}_2[t] \cong H\mathbb{F}_2^* \Sigma^t S$. Recall Ext^0 is naturally isomorphic to Hom.

Proposition 2.1. *1.* $\mathrm{Hom}_{\mathcal{A}}(\mathbb{F}_2, \mathbb{F}_2) \cong \mathbb{F}_2$*, and* $\mathrm{Hom}_{\mathcal{A}}(\mathbb{F}_2, \mathbb{F}_2[t]) \cong 0$ *for* $t \neq 0$.

2. $\mathrm{Ext}^1_{\mathcal{A}}(\mathbb{F}_2, \mathbb{F}_2[t]) \cong 0$ *if* $t \neq 2^j$*. For* $t = 2^j$*, we have* $\mathrm{Ext}^1_{\mathcal{A}}(\mathbb{F}_2, \mathbb{F}_2[t]) \cong \mathbb{F}_2$.

Proof. Let $\overline{\mathcal{A}}$ denote the elements of positive degree of $\mathcal{A}$, so we have a short exact sequence

$$0 \to \overline{\mathcal{A}} \to \mathcal{A} \to \mathbb{F}_2 \to 0.$$

One of the properties of derived functors like Ext is that short exact sequences induce long exact sequences. In this case, we obtain

$$0 \to \mathrm{Hom}_{\mathcal{A}}(\mathbb{F}_2, \mathbb{F}_2[t]) \to \mathrm{Hom}_{\mathcal{A}}(\mathcal{A}, \mathbb{F}_2[t]) \to \mathrm{Hom}_{\mathcal{A}}(\overline{\mathcal{A}}, \mathbb{F}_2[t]) \to \mathrm{Ext}^1_{\mathcal{A}}(\mathbb{F}_2, \mathbb{F}_2[t]) \to \mathrm{Ext}^1_{\mathcal{A}}(\mathcal{A}, \mathbb{F}_2[t]) \cong 0,$$

where $\mathrm{Ext}^1_{\mathcal{A}}(\mathcal{A}, \mathbb{F}_2[t]) \cong 0$ because $\mathcal{A}$ is a projective $\mathcal{A}$ module. For $t \neq 0$, we have $\mathrm{Hom}_{\mathcal{A}}(\mathcal{A}, \mathbb{F}_2[t]) \cong 0$, giving an isomorphism

$$\mathrm{Ext}^1_{\mathcal{A}}(\mathbb{F}_2, \mathbb{F}_2[t]) \cong \mathrm{Hom}_{\mathcal{A}}(\overline{\mathcal{A}}, \mathbb{F}_2[t]). \tag{1}$$

Note we also have an isomorphism $\mathrm{Hom}_{\mathcal{A}}(\mathbb{F}_2, \mathbb{F}_2[t]) \cong 0$ for $t \neq 0$. For $t = 0$, note that

$$\mathrm{Hom}_{\mathcal{A}}(\mathcal{A}, \mathbb{F}_2) \to \mathrm{Hom}_{\mathcal{A}}(\overline{\mathcal{A}}, \mathbb{F}_2)$$

is the map $\mathbb{Z}/2 \to 0$, so we have shown (1). We also have

$$0 \cong \mathrm{Hom}_{\mathcal{A}}(\overline{\mathcal{A}}, \mathbb{F}_2) \to \mathrm{Ext}^1_{\mathcal{A}}(\mathbb{F}_2, \mathbb{F}_2) \to \mathrm{Ext}^1_{\mathcal{A}}(\mathcal{A}, \mathbb{F}_2) \cong 0$$

giving that $\mathrm{Ext}^0_{\mathcal{A}}(\mathbb{F}_2, \mathbb{F}_2) = 0$ as claimed.

Returning to the isomorphism (1), note that

$$\begin{aligned}\mathrm{Ext}^1_{\mathcal{A}}(\mathbb{F}_2, \mathbb{F}_2[t]) &\cong \mathrm{Hom}_{\mathcal{A}}(\overline{\mathcal{A}}, \mathbb{F}_2[t]) \\ &\cong \mathrm{Hom}_{\mathbb{F}_2}(\mathbb{F}_2 \otimes_{\mathcal{A}} \overline{\mathcal{A}}, \mathbb{F}_2[t]) \\ &\cong \mathrm{Hom}_{\mathbb{F}_2}(\overline{\mathcal{A}}/\overline{\mathcal{A}}^2, \mathbb{F}_2[t])\end{aligned}$$

where the second isomorphism uses the change of rings $\mathcal{A} \to \mathbb{F}_2$. We claim that $\overline{\mathcal{A}}/\overline{\mathcal{A}}^2$ is 0 except in dimension 2^i where it is generated by Sq^i, showing the proposition. To see this claim, suppose i is not a power of 2. Then $i = a + 2^k$ with $0 < a < 2^k$. Let $b = 2^k$. Then the Adem relations imply

$$\mathrm{Sq}^a \mathrm{Sq}^b = \binom{b-1}{a} \mathrm{Sq}^{a+b} + \Sigma_{c>0} \binom{b-c-1}{a-2c} \mathrm{Sq}^{a+b-c} \mathrm{Sq}^c.$$

Since b is a power of 2, it follows that $\binom{b-1}{a}$ is odd. Thus $\mathrm{Sq}^i = \mathrm{Sq}^{a+b}$ is in $\overline{\mathcal{A}}^2$.

Now suppose that i is a power of 2. We wish to show that Sq^i is not in $\overline{\mathcal{A}}^2$. Choose an isomorphism $H^*(\mathbb{RP}^\infty, \mathbb{F}_2) \cong \mathbb{F}_2[x]$. It follows from (2) that $\mathrm{Sq}\, x = x + x^2$. By (1), we have $\mathrm{Sq}(x^i) = (x + x^2)^i$. Since i is a power of 2, we have the equality $(x + x^2)^i = x^i + x^{2i}$ modulo 2. Thus $\mathrm{Sq}^j(x^i) = 0$ for $0 < j < i$, and $\mathrm{Sq}^i(x^i)$ is non-zero. It follows that Sq^i can not be written as a linear combination of $\mathrm{Sq}^a \mathrm{Sq}^b$ terms unless $\{a, b\} = \{0, i\}$. This implies that Sq^i is not in $\overline{\mathcal{A}}^2$.

□

Let h_j denote the non-zero element of $\mathrm{Ext}^1_{\mathcal{A}}(\mathbb{F}_2, H\mathbb{F}_2^* \Sigma^{2^j} S)$. Here is a picture of the E^2-page. Just like Hom has a non-commutative ring structure, so does Ext and there are some products labeled in the figure below. On the 2-line $h_i h_j = h_j h_i$.

h_0^2	0	h_1^2	$h_2 h_0$	0	0	h_2^2	$h_3 h_0$	$h_3 h_1$
$h_0 = 2$	$h_1 = \eta$	0	$h_2 = \nu$	0	0	0	$h_3 = \sigma$	0
1	0	0	0	0	0	0	0	0

Figure 1: The first three rows, nine columns of the E_2-page of the Adams spectral sequence. s is on the vertical axis, and $t - s$ on the horizontal axis.

It is a result originally due to Adams that the differential d_2 has the following behavior on the line $s = 2$.

Theorem 2.2. *For $i \geq 4$,*

$$d_2 h_i = h_{i-1} h_0^2 \neq 0.$$

There is a clever inductive argument in [W, Thm 3.6] proving this theorem.

3 Hopf Invariant

Let $C(f)$ denote the topological space which is the mapping cone of $f : S^m \to S^n$ in topological spaces. Assume $m < n$ so that the degree of f is 0. Note that $\tilde{H}^*(C(f)) \cong \mathbb{Z}a \oplus \mathbb{Z}b$ with the a of degree n and b of degree $m+1$. When $m = 2n-1$, define the *Hopf invariant* $H(f) \in \mathbb{Z}$ of f by the formula

$$a^2 = H(f)b.$$

We could exchange b with $-b$, so to have a well-defined sign for $H(f)$, we specify that b maps to a fixed generator of $\tilde{H}^m(S^m) \cong \tilde{H}^m(D^m, \partial D^m)$ under the map $\tilde{H}^m(C(f)) \cong \tilde{H}^m(C(f), S^n) \to \tilde{H}^m(D^m, \partial D^m)$ induced by the map $(D^m, \partial D^m) \to (C(f), S^n)$.

Theorem 3.1. *(Adams) There is a map $f : S^{2n-1} \to S^n$ of Hopf invariant one if and only if $n = 2, 4, 8$.*

There is a list of interesting consequences of this theorem in Hatcher [H, p. 248].

Proof. The division algebra structures on the complex numbers $\mathbb{C}$, the quaternions $\mathbb{H}$, and the octonions $\mathbb{O}$, give maps

$$\eta : S^3 \subset \mathbb{C}^2 - \{0\} \to \mathbb{CP}^1 \cong S^2$$
$$\nu : S^7 \subset \mathbb{H}^2 - \{0\} \to \mathbb{HP}^1 \cong S^4$$
$$\sigma : S^{15} \subset \mathbb{O}^2 - \{0\} \to \mathbb{OP}^1 \cong S^8$$

whose mapping cones are $\mathbb{CP}^2$, $\mathbb{HP}^2$, and $\mathbb{OP}^2$ respectively. The cup product structure on projective spaces is polynomial, so these maps have Hopf invariant 1.

Take $n \neq 2, 4, 8$, and $f : S^{2n-1} \to S^n$. We wish to show that f does not have Hopf invariant one. Let C denote the mapping cone of f. Note that Sq^n

takes the n-dimensional class in $H^*(C, \mathbb{F}_2)$ to the $2n$-dimensional class. Thus the extension

$$0 \to H^*(S^{2n}, \mathbb{F}_2) \to H^*(C, \mathbb{F}_2) \to H^*(S^n, \mathbb{F}_2) \to 0$$

does not split as a short exact sequence of $\mathcal{A}$-modules. Therefore, f represents a non-trivial class in $\mathrm{Ext}^1_{\mathcal{A}}(\mathbb{F}_2[n], \mathbb{F}_2[2n]) \cong \mathrm{Ext}^1_{\mathcal{A}}(\mathbb{F}_2, \mathbb{F}_2[n])$. By Proposition 2.1, it follows that $n = 2^i$.

Since f represents an element of $\pi_{n-1}S$, it follows that all the differentials in the Adams spectral sequence vanish on the element of $\mathrm{Ext}^1_{\mathcal{A}}(\mathbb{F}_2, \mathbb{F}_2[n])$ corresponding to f. By Theorem 2.2, it follows that $i < 4$. □

4 Adams-Atiyah proof of the Hopf invariant one theorem

Here is an alternate proof of the "only if" direction in Theorem 3.1 due to Atiyah in the case where n is even. (Recall that our calculation of $\mathrm{Ext}^1_{\mathcal{A}}(\mathbb{F}_2, \mathbb{F}_2[n])$ implies that n is a power of 2.)

Let n be even and let $f : S^{2n-1} \to S^n$ be a map of Hopf invariant one. Let $C(f)$ be the mapping cone. We obtain an extension

$$0 \to \tilde{K}^{-1}(S^{2n-1}) \cong \tilde{K}^0(S^{2n}) \to \tilde{K}^0(C(f)) \to \tilde{K}^0 S^n \to 0, \tag{2}$$

since $\tilde{K}^{-1} S^n \cong 0$ and $\tilde{K}^{-1}(S^{2n}) \cong 0$.

Let x be a generator of $\tilde{K}^0 S^n \cong \mathbb{Z}$, and let y be a generator of $\tilde{K}^0(S^{2n}) \cong \mathbb{Z}$. It follows that there is an integer $h(f)$ such that $x^2 = h(f)y$. Recall the ring homomorphism $\mathrm{ch} : \tilde{K}^0(X) \to \tilde{H}^0(X, \mathbb{Q})$ from Lecture 1.3. By Proposition 1.3 of Lecture 1.3, we have that the Hopf invariant $H(f)$ equals $h(f)$ potentially after swapping y for $-y$. Thus we have that $x^2 = y$.

Recall that the Adams operations ψ^k act on all the groups in (2). By Lemma 1.1 of Lecture 17, we have that ψ^k acts on $\tilde{K}^0(S^{2n})$ by multiplication by k^n. Thus $\psi^k y = k^n y$. Let $m = n/2$. We also have ψ^k acts on $\tilde{K}^0(S^n)$ by multiplication by k^m. It follows that $\psi^k(x) = k^m x + c_k y$ for some $c_k \in \mathbb{Z}$. By Lecture 16 Theorem 1.1 (2), it follows that $\psi^2(x) \cong x^2 \mod 2$. Thus $\psi^2 x \cong y \mod 2$, whence c_2 is odd.

By Lecture 16 Theorem 1.1 (4), we have $\psi^3\psi^2 x = \psi^2\psi^3 x$. This implies

$$\psi^3(2^m x + c_2 y) = \psi^2(3^m x + c_3 y)$$

$$3^m 2^m x + 2^m c_3 y + c_2 3^{2m} y = 2^m 3^m x + c_2 3^m y + 2^{2m} c_3 y.$$

Thus

$$c_3(2^m - 2^{2m}) = c_2(3^m - 3^{2m}).$$

Since c_2 is odd, we must have that $3^m - 3^{2m}$ is divisible by 2^m. It is a fact from number theory that this only happens when $m = 1, 2, 4$.

5 Useful websites

`http://ext-chart.org`

`http://www.math.wayne.edu/~rrb/cohom/index.html`

`http://www.nullhomotopie.de/charts/index.html`

References

[G] Paul Goerss, *The Adams-Novikov Spectral Sequence and Homotopy Groups of Spheres.*

[H] Allen Hatcher, *Algebraic Topology.*

[Ma] H.R. Margolis *Spectra and the Steenrod Algebra* North-Holland Mathematical Library, volume 29, 1983.

[M] Haynes Miller, *The Adams Spectral Sequence, etc. (course notes).*

[MT] Robert Mosher and Martin Tangora, *Cohomology Operations and Applications in Homotopy Theory*, Dover, 1968, 2008.

[W] Wang, John *On the cohomology of the mod 2 Steenrod algebra*, Topology, 10, 1971, p.53-65.