Lecture 22: Adams Spectral Sequence for $H\mathbb{Z}/p$

4/10/15-4/17/15

1 A brief recall on Ext

We review the definition of Ext from homological algebra. Ext is the derived functor of Hom. Let R be an (ordinary) ring. There is a model structure on the category of chain complexes of modules over R where the weak equivalences are maps of chain complexes which are isomorphisms on homology.

Given a short exact sequence

$$0 \to M_1 \to M_2 \to M_3 \to 0$$

of R-modules, applying $\mathrm{Hom}(P, -)$ produces a left exact sequence

$$0 \to \mathrm{Hom}(P, M_1) \to \mathrm{Hom}(P, M_2) \to \mathrm{Hom}(P, M_3).$$

A module P is *projective* if this sequence is always exact. Equivalently, P is projective if and only if if it is a retract of a free module. Let M_* be a chain complex of R-modules. A *projective resolution* is a chain complex P_* of projective modules and a map $P_* \to M_*$ which is an isomorphism on homology. This is a cofibrant replacement, i.e. a factorization of $0 \to M_*$ as a cofibration followed by a trivial fibration is $0 \to P_* \to M_*$. There is a functor Hom that takes two chain complexes M and N and produces a chain complex $\mathrm{Hom}(M_*, N_*)$ defined as follows. The degree n module, are the $\prod_* \mathrm{Hom}(M_*, N_{*+n})$. To give the differential, we should fix conventions about degrees. Say that our differentials have degree -1. Then there are two maps

$$\prod_* \mathrm{Hom}(M_*, N_{*+n}) \to \prod_* \mathrm{Hom}(M_*, N_{*+n-1}).$$

The first takes $\prod f_n$ to $d_N f_n$. The second takes $\prod f_n$ to $f_{n-1} d_M$. The differential on $\mathrm{Hom}(M_*, N_*)$ is the difference. To get a feel for this definition, note that the kernel of the differential is exactly the chain maps. Given a chain complex N_*, we derive the functor $\mathrm{Hom}(-, N_*)$ to the functor $\mathrm{Ext}(M_*, N_*) := \mathrm{Hom}(P_*, N_*)$.

Frequently, one sees this definition as a sequence of modules $\operatorname{Ext}^i(M,N)$ associated to two modules M and N. To recover this, view M and N as chain complexes with a single non-zero module in degree 0, and define $\operatorname{Ext}^i(M,N) = H_i \operatorname{Ext}(M_*, N_*)$.

Ext^1 classifies extensions or short exact sequences – see [L, XX Ex 27 p 831].

2 Adams resolutions and towers

Let $\mathcal{A} = H\mathbb{Z}/p^* H\mathbb{Z}/p$ denote the mod-p Steenrod algebra. In L21, we considered the generalized degree homomorphism

$$[X,Y] \to \operatorname{Hom}_{\mathcal{A}}(H\mathbb{Z}/p^* Y, H\mathbb{Z}/p^* X), \tag{1}$$

commented that it isn't usually an isomorphism, took elements of the kernel etc. However, there are certain Y for which (1) is an isomorphism.

For a graded $\mathbb{F}_p$-vector space $V = \oplus_{*\in\mathbb{Z}} V_*$, let $HV = \vee_n \Sigma^n HV_n$. We will suppose that V_n is finite dimensional for all n and is 0 for n sufficiently small. Spectra of the form HV are called *mod p Eilenberg-MacLane spectra.*

We are going to resolve spectra by mod p Eilenberg-MacLane spectra. This is analogous to replacing a module by a projective module. We first prove the properties of mod p Eilenberg-MacLane spectra that will show that such a resolution is an "improvement," for example because (1) is an isomorphism.

2.1 (1) is an isomorphism for mod p Eilenberg-MacLane spectra

In this subsection, we prove some properties of mod p Eilenberg-MacLane spectra.

The homotopy groups of HV are $\pi_m HV \cong V_m$, because generalized homology theories like π_m take wedge sums to $\oplus$. (In more detail, this argument is as follows. Note that $HV \cong \operatorname{colim}_N \vee_{n<N} \Sigma^n HV_n$. π_m commutes with colim_N, because S^m is compact. Also, $\pi_m(\Sigma^n HV_n) \cong V_n$ for $n = m$ and 0 otherwise. Also, for a finite wedge-sum, the homotopy groups are the sum (by the split long exact sequence). Thus $\pi_m \vee_{n<N} \Sigma^n HV_n \cong V_m$ for $N > m$, showing the claim.)

We will also need to use the fact:

Lemma 2.1. *The Steenrod algebra is concentrated in non-negative degrees, i.e.* $\mathcal{A}^n = 0$ *for* $n < 0$.

Proof. An element a of $\mathcal{A}^n$ gives a natural transformation $t : \mathrm{H}^i(-;\mathbb{Z}/p) \to \mathrm{H}^{i+n}(-;\mathbb{Z}/p)$. By representability of cohomology and Yoneda's lemma, such natural transformations are in bijection with $\mathrm{H}^{i+n}(B(i,\mathbb{Z}/p);\mathbb{Z}/p)$. Then $\mathrm{H}^{i+n}(B(i,\mathbb{Z}/p);\mathbb{Z}/p) \cong 0$ for $n < 0$ because $B(i,\mathbb{Z}/p)$ is $(i-1)$-connected by the Hurewicz theorem. Thus $t = 0$, and it follows that $a = 0$. □

Lemma 2.2. *Suppose that* V_n *is finite dimensional for all* n *and is* 0 *for* n *sufficiently small. Then* $H\mathbb{F}_p^* HV \cong \mathcal{A} \otimes V^\star$, *where* $V^\star$ *is the graded vector space with* $V_*^\star = \mathrm{Hom}(V_*, \mathbb{F}_p)$.

Proof. There is a natural map $H\mathbb{F}_p^* HV \to \mathrm{Hom}(\pi_* HV, \pi_0 H\mathbb{F}_p) \cong V_*^\star$. Let B_n be a basis of V_n. B_n gives a splitting of this map: in degree n, this splitting sends an element of the dual basis to the identity function in $H\mathbb{F}_p^0 H\mathbb{F}_p$ which is then mapped $H\mathbb{F}_p^0 H\mathbb{F}_2 \to H\mathbb{F}_p^n HV$ by the inclusion of the $\Sigma^n H\mathbb{F}_p$ corresponding to the basis element. Since $H\mathbb{F}_p^* HV$ is a module over $\mathcal{A}$, we obtain a map $\mathcal{A} \otimes V^\star \to H\mathbb{F}_p^* HV$. We claim this map is an isomorphism.

$H\mathbb{F}_p^n HV \cong \pi_{-n} F(HV, H\mathbb{F}_p)$. We have $HV = \vee_n \vee_{B_n} \Sigma^n H\mathbb{F}_p$. Functions out of a colimit are the limit of the functions. Thus $F(\vee_n \vee_{B_n} \Sigma^n H\mathbb{F}_p, H\mathbb{F}_p) \cong \prod_{n,B_n} F(\Sigma^n H\mathbb{F}_p, H\mathbb{F}_p)$. Thus

$$H\mathbb{F}_p^n HV \cong \pi_{-n} \prod_{m,B_m} F(\Sigma^m H\mathbb{F}_p, H\mathbb{F}_p) \cong \prod_{m,B_m} \pi_{m-n} F(H\mathbb{F}_p, H\mathbb{F}^p) \cong \prod_{m,B_m} \mathcal{A}^{n-m}.$$

By Lemma 2.1, $\mathcal{A}^{n-m}$ is only non-zero when $n - m \geq 0$. Thus, the product $\prod_{m,B_m} \mathcal{A}^{n-m}$ is a finite product, and can be written $\oplus_m V_m^\star \otimes \mathcal{A}$. □

A spectrum X is called *bounded below* if there exists an integer r such that $\pi_* X = 0$ for all $* < r$.

Proposition 2.3. *Assume that* X *is a spectrum which is bounded below and has finitely generated* $\pi_* X$ *for all* $*$, *and that* V_* *is finite dimensional for all* $*$. *The generalized degree homomorphism* (1) *is an isomorphism for* $Y = HV$ *a mod* p *Eilenberg-MacLane spectrum.*

Proof. We claim that $[X, HV] \cong \mathrm{Hom}_{\mathbb{F}_p}((H\mathbb{F}_p)_* X, V)$. Given a map $X \to HV$, we may smash with $H\mathbb{F}_p$, and compose with the map $H\mathbb{F}_p \wedge HV \to HV$ coming from the $\mathbb{F}_p$-module structure on V. We obtain a map $H\mathbb{F}_p \wedge X \to HV$. Applying π_* gives a map $(H\mathbb{F}_p)_* X \to V$. This defines a natural transformation

$$[X, HV] \to \mathrm{Hom}_{\mathbb{F}_p}((H\mathbb{F}_p)_* X, V),$$

which is an isomorphism when X is a sphere. Both sides define generalized cohomology theories, so the natural transformation must be a natural isomorphism.

By Lemma 2.2, $H\mathbb{F}_p^* HV \cong \mathcal{A} \otimes V^\star$. This allows the following calculation.

$$\mathrm{Hom}_{\mathcal{A}}((H\mathbb{F}_p)^* HV, (H\mathbb{F}_p)^* X) \cong \mathrm{Hom}_{\mathbb{F}_p}(V^\star, (H\mathbb{F}_p)^* X) \cong \prod_n (V_n \otimes H\mathbb{F}_p^n X) \cong$$
$$\prod_n V_n \otimes \mathrm{Hom}_{\mathbb{F}_p}((H\mathbb{F}_p)_* X, \mathbb{F}_p)) \cong \prod_n \mathrm{Hom}_{\mathbb{F}_p}((H\mathbb{F}_p)_* X, V_n)) \cong \mathrm{Hom}_{\mathbb{F}_p}((H\mathbb{F}_p)_* X, V)$$
$$\cong [X, HV]$$

□

2.2 Adams resolutions

Assumption 2.4. *All spectra considered will be bounded below and have finitely generated homotopy groups in all dimensions.*

Definition 2.5. *Given a spectrum Y, an* Adams resolution *for Y is a chain complex of spectra*

$$Y \to W_0 \to W_1 \to W_2 \to \dots$$

such that each W_i is a mod p Eilenberg-MacLane spectrum and such that for all mod p Eilenberg-MacLane spectra HV, the chain complex of vector spaces

$$[Y, HV] \leftarrow [W_0, HV] \leftarrow [W_1, HV] \leftarrow [W_2, HV] \leftarrow$$

is exact.

Given a free resolution

$$0 \leftarrow H\mathbb{F}_p^* Y \leftarrow P_0 \leftarrow P_1 \leftarrow P_2 \leftarrow \dots$$

of $H\mathbb{F}_p^* Y$ as a $\mathcal{A}$-module, we can construct an Adams resolution as follows. P_i is isomorphic to $\mathcal{A} \otimes V$ for a graded vector space V, which is bounded below and finite dimensional in each dimension. This implies that $(V^\star)^\star \cong V$. Thus $H\mathbb{F}_p^* HV^\star \cong P_i$. Let $W_i = HV^\star$. By Proposition 2.3, there is unique element of $[W_i, W_{i+1}]$ corresponding to the map of $\mathcal{A}$-modules $P_i \leftarrow P_{i+1}$. Let the map $W_i \to W_{i+1}$ be in this homotopy class. Similarly, let $Y \to W_0$ correspond to $H\mathbb{F}_p^* Y \leftarrow P_0$ via Proposition 2.3. We then have $Y \to W_0 \to W_1 \to \dots$.

By construction, this sequence gives an exact sequence of vector spaces after applying $[-, \Sigma^n H\mathbb{F}_p]$. In the proof of Proposition 2.3, we showed that $[X, HV] \cong \mathrm{Hom}_{\mathbb{F}_p}((H\mathbb{F}_p)_* X, V)$. It follows that $Y \to W_0 \to W_1 \to \ldots$ is an Adams resolution as claimed.

Remark 2.6. *In fact, a projective resolution of $H\mathbb{F}_p^* Y$ gives rise to an Adams resolution. Any projective module P_i is a retract of a free module F_i. Choose W_i' with $\mathbb{F}_p$-cohomology F_i. Let $r : F_i \to P_i$ and $i : P_i \to F_i$ be the retraction. Then we can construct a map of spectra $W_i' \to W_i'$ whose corresponding map on cohomology is ir. Taking the colimit $\ldots \to W_i' \to W_i' \to \ldots$ produces a spectrum W_i with cohomology P_i. The rest of the argument is identical.*

There is also a uniqueness statement for Adams resolutions, but we're going to skip it.

2.3 Adams tower

The data of an Adams resolution is equivalent to the following data, which is called an Adams tower.

Definition 2.7. *An* Adams tower *of a spectrum Y is*

$$\begin{array}{ccccccc} Y = Y_0 & \xleftarrow[i_1]{} & Y_1 & \xleftarrow[i_2]{} & Y_2 & \longleftarrow & \ldots \\ \downarrow & & \downarrow & & \downarrow & & \\ W_0 & & \Sigma^{-1} W_1 & & \Sigma^{-2} W_2 & & \end{array}$$

such that

$$\Sigma^s Y_{s+1} \xrightarrow{\Sigma^s i_{s+1}} \Sigma^s Y_s \xrightarrow{k_s} W_s \tag{2}$$

is a cofiber sequence, W_i is a mod p Eilenberg-MacLane spectrum, and $H\mathbb{F}_p^* k_s$ is surjective.

We won't do both ways of this equivalence, but given an Adams tower, we can construct an Adams resolution as follows. The cofiber sequence (2) gives a map $W_s \to \Sigma^{s+1} Y_{s+1}$. Since Y_{s+1} maps to $\Sigma^{-(s+1)} W_{s+1}$, we may compose the $(s+1)$st suspension of this map with $W_s \to \Sigma^{s+1} Y_{s+1}$, obtaining a map $W_s \to W_{s+1}$. To see that the resulting

$$Y \to W_0 \to W_1 \to \ldots$$

is indeed an Adams resolution, note that the assumption that $H\mathbb{F}_p^* k_s$ is surjective combined with the long exact sequence on $H\mathbb{F}_p^*$ applied to the cofiber sequence (2) gives short exact sequences

$$0 \leftarrow H\mathbb{F}_p^* \Sigma^s Y_s \leftarrow H\mathbb{F}_p^* W_i \leftarrow H\mathbb{F}_p^* \Sigma^{s+1} Y_{s+1} \leftarrow 0.$$

Combining the sequence for $i-1$,i, and $i+1$ gives exactness at W_i for the mod p Eilenberg-MacLane spectrum $\Sigma^* H\mathbb{F}_p$. The case of general HV follows using $[X, HV] \cong \mathrm{Hom}_{\mathbb{F}_p}((H\mathbb{F}_p)_* X, V)$, which was shown in the proof of Proposition 2.3.

2.4 Adams spectral sequence

Take spectra Y and X satisfying the assumption. Choose an Adams tower for Y. Applying $[\Sigma^t X, -]$ to the cofiber sequences (2) yields long exact sequences

$$\ldots \to [\Sigma^t X, \Sigma^s Y_{s+1}] \to [\Sigma^t X, \Sigma^s Y_s] \to [\Sigma^t X, W_s] \to [\Sigma^{t-1} X, \Sigma^s Y_{s+1}] \to \ldots.$$

For notational convenience, define $Y_s = Y_0$ for $s < 0$.

These long exact sequences determine an exact couple

$$\begin{array}{ccc} \oplus_{s,t}[\Sigma^t X, \Sigma^s Y_{s+1}] & \longrightarrow & \oplus_{s,t}[\Sigma^t X, \Sigma^s Y_s]\ , \\ & \nwarrow \quad \swarrow & \\ & \oplus_{s,t}[\Sigma^t X, W_s] & \end{array}$$

which in turn determines a spectral sequence $E_r^{t,s}$ for $r = 1, 2, \ldots$ where $E^1_{t,s} = [\Sigma^t X, W_s]$ and $d_r : E_r^{t,s} \to E_r^{t-1+r,s+r}$. To see that this is the bidegree of the differential d_r, recall that d_r is obtained from the commutative diagram

$$\begin{array}{ccccc} & [\Sigma^{t-1} X, \Sigma^s Y_{s+r}] \ldots \longrightarrow [\Sigma^{t-1} X, \Sigma^s Y_{s+2}] \longrightarrow [\Sigma^{t-1} X, \Sigma^s Y_{s+1}] & \\ \swarrow & & \nwarrow \\ [\Sigma^{t-1} X, \Sigma^{-r} W_{s+r}] & & [\Sigma^t X, W_s] \end{array}$$

by applying the right-most arrow, then choosing lifts along the $r-1$ arrows along the top and then applying the left-most arrow.

This is the *Adams spectral sequence* for $H\mathbb{F}_p$ and $[X, Y]_*$.

2.5 E_2-page

By Proposition 2.3, we may compute $[\Sigma^t X, W_s] \cong \mathrm{Hom}_{\mathcal{A}}(H\mathbb{F}_p^* W_s, H\mathbb{F}_p^* \Sigma^t X)$.

The differential $d_1 : [\Sigma^t X, W_s] \to [\Sigma^{t-1} X, \Sigma^{-1} W_{s+1}] \cong [\Sigma^t X, W_{s+1}]$ is obtained by applying applying $[\Sigma^t X, -]$ to the map $W_s \to W_{s+1}$ coming from the Adams resolution associated to our Adams tower. (This is by construction.)

Therefore we have that $E^2_{t,s}$ is the sth homology of the chain complex of vector spaces obtained by applying $\mathrm{Hom}_{\mathcal{A}}(-, H\mathbb{F}_p^*\Sigma^t X)$ to the exact sequence

$$0 \leftarrow H\mathbb{F}_p^*Y \leftarrow H\mathbb{F}_p^*W_1 \leftarrow H\mathbb{F}_p^*W_2 \leftarrow .$$

By definition of Ext, it follows that $E^2_{t,s} \cong \mathrm{Ext}^s_{\mathcal{A}}(H\mathbb{F}_p^*Y, H\mathbb{F}_p^*\Sigma^t X)$

2.6 Mechanics of the spectral sequence

While it's true that we've seen the machine that takes an exact couple and gives a spectral sequence, I think it would be helpful to take a look at this machine while it's producing the Adams spectral sequence. Besides, we'd like to connect the Adams spectral sequence to the discussion in Lecture 21.

Start with a map $f : \Sigma^t X \to Y$. Form the composite $k_0 f : \Sigma^t X \to Y \to W_0$. By Proposition 2.3, the homotopy class of the composite is the same information as $H\mathbb{F}_p^*(k_0 f)$ in $\mathrm{Hom}_{\mathcal{A}}(H\mathbb{F}_p^*W_0, H\mathbb{F}_p^*\Sigma^t X)$. Since $H\mathbb{F}_p^*W_0 \to H\mathbb{F}_p^*Y$ is a surjection, we have that $H\mathbb{F}_p^*(k_0 f)$ is 0 if and only if $H\mathbb{F}_p^*f$ is 0.

The $H\mathbb{F}_p^*$-degrees of maps $[\Sigma^t X, Y]$ are going to be recorded in $E^\infty_{t,0}$.

We've just seen that these degrees are the image of $[\Sigma^t X, Y] \to [\Sigma^t X, W_0]$. $E^1_{t,0} = \mathrm{Hom}_{\mathcal{A}}(H\mathbb{F}_p^*W_0, H\mathbb{F}_p^*X) \cong [\Sigma^t X, W_0]$ is a first approximation.The second approximation $E^2_{t,0} = \mathrm{Hom}_{\mathcal{A}}(H\mathbb{F}_p^*Y, H\mathbb{F}_p^*X)$ is better. From the cofiber sequence

$$Y_1 \to Y_0 = Y \to W_0$$

and its associated long exact sequence, we know what the possible degrees are. Namely, the LES means the degrees (which are this image of $[\Sigma^t X, Y] \to [\Sigma^t X, W_0]$) are also the elements of the kernel of

$$[\Sigma^t X, W_0] \to [\Sigma^t X, \Sigma Y_1]. \tag{3}$$

This kernel is not directly recorded in our spectral sequence. It couldn't really be put into the spectral sequence in a useful manner because Y_1 is not one of our mod p Eilenberg-MacLane spaces, and we don't have a good handle on $[\Sigma^t X, \Sigma Y_1]$. What the spectral sequence does is say " well, we can not compute the kernel of (3), but we can make better and better approximations." First of all, anything in the kernel of (3) is also in the kernel of $[\Sigma^t X, W_0] \to [\Sigma^t X, \Sigma\Sigma^{-1} W_1]$, which is precisely the kernel of d_1. If g in $[\Sigma^t X, W_0]$ is in the kernel of d_1, this means that the image of g in $[\Sigma^t X, \Sigma Y_1]$ is in the kernel of

$$[\Sigma^t X, \Sigma Y_1] \to [\Sigma^t X, \Sigma\Sigma^{-1} W_1]. \tag{4}$$

From the cofiber sequence $\Sigma Y_2 \to \Sigma Y_1 \to W_1$, the kernel of (4) is the image of $[\Sigma^t X, \Sigma Y_2] \to [\Sigma^t X, \Sigma Y_1]$. Therefore g can be lifted to a map $g_2 : \Sigma^t X \to \Sigma Y_2$. Note that g is really in the kernel of (3) if and only if g_2 is null.

In the same way that we did not have a direct handle on $[\Sigma^t X, \Sigma Y_1]$, we do not have a direct handle on $[\Sigma^1 X, \Sigma Y_2]$. But, we have a map $\Sigma Y_2 \to \Sigma^{-1} W_2$. If g_2 is null, then its image under

$$[\Sigma^1 X, \Sigma Y_2] \to [\Sigma^t X, \Sigma^{-1} W_2] \tag{5}$$

is null as well. The image of g_2 under (5) is the value of the second differential $d_2 g$ of the Adams spectral sequence on g. We can do the same process with g_2 and obtain g_3 and $d_3 g$, and continue on in this way. This produces a lot of necessary conditions on an element of $E^1_{t,0}$ to be a degree of a map $[\Sigma^t X, Y]$. In fact, if X is a finite spectrum this is an if and only if. This phenomenon is part of the statement that the spectral sequence converges.

We can move up to $E^\infty_{t,1}$.

If two maps $f, g : \Sigma^t X \to Y$ have the same $H\mathbb{F}_p^*$-degree, then their difference $f - g$ has degree 0.

The $H\mathbb{F}_p^*$-extensions of maps $[\Sigma^t X, Y]$ of degree 0 are going to be recorded in $E^\infty_{t+1,1}$.

For brevity, let's define $h = f - g$, so h is a map $\Sigma^t X \to Y$ which has degree 0. As above, h having degree 0 is equivalent to the image of h under $[\Sigma^t X, Y] \to [\Sigma^t X, W_0]$ being 0. Using the cofiber sequence $Y_1 \to Y_0 \to W_0$, we may lift h to a map $\overline{h} : \Sigma^t X \to Y_1$. Composing $\overline{h}$ with the map $\Sigma^{-1} k_1 : Y_1 \to \Sigma^{-1} W_1$ gives the element $\Sigma^{-1} k_1 \overline{h} \in E^1_{t+1,1}$. When we run the spectral sequence we will have differentials leaving $E^r_{t+1,1}$ as above, together with one differential coming in $E^2_{t+1,0} \to E^2_{t+1,1}$, and taking the kernels of the first modulo the image of the second produces an $E^\infty_{t+1,1}$ which contains the extension classes of maps h in $[\Sigma^t X, Y]$. The fact that these necessary conditions are also sufficient when X is finite and under our hypotheses on Y is again a matter of convergence.

Exercise 2.8. *Show that $\Sigma^{-1} k_1 \overline{h}$ is in the kernel of d_1 (really it's in the kernel of all d_i's but this is just an exercise). Combine this with the classification of extensions in [L, XX Ex 27 p 831] to show that $\Sigma^{-1} k_1 \overline{h} \in E^2_{t+1,1} \cong \mathrm{Ext}^1_{\mathcal{A}}(H\mathbb{F}_p^* Y, H\mathbb{F}_p^* \Sigma^{t+1} X)$ classifies the extension*

$$0 \leftarrow H\mathbb{F}_p^* Y \leftarrow H\mathbb{F}_p^* C \leftarrow H\mathbb{F}_p^* \Sigma^{t+1} X \leftarrow 0,$$

resulting from the mapping cone $\Sigma^t X \to Y \to C$ as in Lecture 21.

2.7 Convergence

We're not going to prove convergence statements for the Adams spectral sequence, but here are results.

Associated to Y and a spectrum E, there is a map $Y \to L_E Y$ characterized by the property that it induces an isomorphism on E-homology $E_* Y \xrightarrow{\cong} E_* L_E Y$, and any map $X \to L_E Y$ with X such that $E_* X \cong 0$ is null-homotopic. The spectrum $L_E Y$ is called the *E-localization of* Y. Heuristically, $L_E Y$ is the spectrum which holds everything about Y that can be seen by E and does not hold any information invisible to E. Under the hypothesis (that we've made) that the homotopy groups of Y are finitely generated, then $\pi_*(L_{H\mathbb{F}_p} Y) \cong \pi_* Y \otimes \mathbb{Z}_p$, where $\mathbb{Z}_p$ are the p-adics.

The $H\mathbb{F}_p$-localization of Y can be obtained from any Adams tower as follows. Let X^s be defined to be the cofiber of the map $Y_{s+1} \to Y_0 = Y$ in the Adams tower. The map $Y_{s+1} \to Y_s$ induces a map $Y^s \to Y^{s-1}$. We obtain

$$\ldots \to Y^2 \to Y^1 \to Y^0 \to 0$$

where each space Y^s receives a map from Y. The resulting map $Y \to \varprojlim_s Y^s$ can be shown to be $Y \to L_{H\mathbb{F}_p} Y$.

Define

$$e^{\infty}_{t,s} = \frac{\operatorname{Ker}[\Sigma^{t-s} X, L_{H\mathbb{F}_p} Y] \to [\Sigma^{t-s} X, Y^{s-1}]}{\operatorname{Ker}[\Sigma^{t-s} X, L_{H\mathbb{F}_p} Y] \to [\Sigma^{t-s} X, Y^{s}]}.$$

For degree reasons, for large r, we will have $E^r_{t,s} \subseteq E^{r-1}_{t,s}$, i.e. all the differentials coming into $E_{t,s}$ will eventually vanish. Thus we may define $E^{\infty}_{t,s} = \cap E^r_{t,s}$ where this intersection takes place over sufficiently large r.

There is a natural map $e^{\infty}_{t,s} \to E^{\infty}_{t,s}$, although this requires unwinding a bunch of definitions to see.

Recall that we said that a spectral sequence converges to a graded group $\oplus G_n$ if there is a filtration of each G_n such that $\oplus_{t-s} E^{\infty}_{t,s} \cong \operatorname{Gr} G_{t-s}$, where $\operatorname{Gr} G_{t-s}$ denotes the associated graded of G_{t-s}.

We would like to say that the Adams Spectral Sequence $E^r_{t,s}$ converges to $[\Sigma^{t-s} X, L_{H\mathbb{F}_p} Y]$ with the filtration whose sth filtered piece is $\operatorname{Ker}[\Sigma^{t-s} X, L_{H\mathbb{F}_p} Y] \to [\Sigma^{t-s} X, Y^{s-1}]$. This happens under the following two conditions:

- $e^{\infty}_{t,s} \to E^{\infty}_{t,s}$ is an isomorphism.

- $[\Sigma^t X, L_{H\mathbb{F}_p} Y] = [\Sigma^t X, \varprojlim Y^s] \to \varprojlim [\Sigma^t X, Y^s]$ is an isomorphism.

The second isomorphism happens if $\varprojlim^1 [\Sigma^{t+1} X, Y^s] \cong 0$ for all t,[BK, IX §3 Thm 3.1]. Both conditions happen if $\varprojlim^1 E^r_{t,s} = 0$ [BK, IX §5 5.4].

For $p = 2$, X a finite spectrum, and Y satisfying our hypotheses, one always has convergence [G, 1.4]

$$\mathrm{Ext}^s_{\mathcal{A}}(H\mathbb{F}_2^* Y, H\mathbb{F}_2^* \Sigma^t X) \Rightarrow [\Sigma^{t-s} X, L_{H\mathbb{F}_2} Y] \cong [\Sigma^{t-s} X, Y] \otimes \mathbb{Z}_2.$$

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