Lecture 21: Generalizing degree

4/8/15

Recall that the *degree* of map $S^n \to S^n$ is the $d \in \mathbb{Z}$ such that $\mathrm{H}^*(S^n, \mathbb{Z}) \to \mathrm{H}^*(S^n, \mathbb{Z})$ is multiplication by d. Given a map $f : Y \to X$ and a spectrum E, consider the induced map

$$E^*(f) : E^*(X) \to E^*(Y).$$

We get natural maps generalizing the degree

$$\pi_t F(Y, X) \to \mathrm{Hom}(E^*(X), E^*(\Sigma^t Y)).$$

For example, when Y is the sphere, this is

$$\pi_t X \to \mathrm{Hom}(E^* X, E * S^t).$$

This map is usually not an isomorphism.

Suppose that $f : Y \to X$ is in the kernel. Form the mapping cone of f, so we have a cofiber sequence $Y \to X \to C(f)$. There is an associated long exact sequence in cohomology

$$\ldots \to E^{n-1}X \stackrel{E^{n-1}(f)}{\to} E^{n-1}Y \to E^n C(f) \to E^n X \stackrel{E^n(f)}{\to} E^n(X) \to \ldots.$$

Since $E^* f = 0$, this long exact sequence splits up into short exact sequences

$$0 \to E^{n-1}Y \to E^n C(f) \to E^n X \to 0,$$

i.e., *extensions*, and they are classified by a group $\mathrm{Ext}^1(E^n X, E^{n-1}Y)$.

So, given a map f, if it is degree 0, we could still hope to detect it as a non-zero element of $\mathrm{Ext}^1(E^n X, E^{n-1}Y)$ classifying extensions.

Example 1.1. *Recall that $\tilde{K}^0(S^{2n}) \cong \mathbb{Z}$ with Adams operations ψ^k acting by multiplication by k^n (Lemma 1.1 Lecture 17) and $\tilde{K}^0(S^{2n-1}) = 0$. Given a map $f : S^{2m-1} \to S^{2n}$, the induced map on K-theory is 0. We have an extension*

$$0 \to \tilde{K}^0 S^{2m} \to \tilde{K}^0 C(f) \to \tilde{K}^0 S^{2n} \to 0.$$

As an extension of abelian groups this must be

$$0 \to \mathbb{Z} \to \mathbb{Z} \oplus \mathbb{Z} \to \mathbb{Z} \to 0,$$

which is the 0*-extension, but we also have Adams operations. To see that this gives an interesting invariant, let* a *be the generator of* $\tilde{K}^0 S^{2m}$ *and let* b *be some element of* $\tilde{K}^0 C(f)$ *mapping to a generator of* $\tilde{K}^0 S^{2n}$*. Then* $\psi^k b$ *must map to* k^n *times the image of* b*. Thus* $\psi^k b = k^n b + ca$*, where* $c \in \mathbb{Z}$*. This* c *gives information about* f*.*

For example, let $k = 2$*, and let* $m = 2n$*, so* f *is a map* $f : S^{4n-1} \to S^{2n}$*, and* $C(f)$ *is the union of a* $4n$ *cell and* S^{2n}*. Then* $b^2 = H(f)a$ *for some* $H(f)$ *in* $\mathbb{Z}$*, called the Hopf invariant. Since* $\psi^2 b \cong b^2 \mod 2$*, we have that* $c \bmod 2$ *is a mod-2* K*-theoretic Hopf invariant.*

We should be more precise about forming a group classifying extensions with the appropriate amount of structure. Supposing we have done this, it could still be the case that our map f had 0 degree, and determined 0 in Ext^1. We then would wish to construct an element of Ext^2 that detected f and so on.

Our next goal is the *Adams spectral sequence*, and it is a tool for studying the homotopy of X, or $F(Y, X)$, using homology. For $E = H\mathbb{Z}/p$, this gives a spectral sequence

$$E_2^{s,t} = \mathrm{Ext}^s_{\mathcal{A}}(\mathrm{H}^*(X; \mathbb{Z}/p), \mathrm{H}^*(S^t; \mathbb{Z}/p)) \Rightarrow \pi_{t-s} X \otimes \mathbb{Z}_p,$$

$$d_r : E_r^{s,t} \to E^{s+r,t+r-1}$$

under certain assumptions on X.

Notice the $\mathcal{A}$ on the right hand side. $\mathcal{A}$ is called the *Steenrod algebra* and it arises as follows. Let $E = H\mathbb{Z}/p$. Let $f : X \to Y$ be a map, and consider the associated map $E^*(f) : E^*(Y) \to E^*(X)$ as before. Recall that $E^*(f)$ is defined by associating to a map $\sigma : \Sigma^{-*} Y \to E$ in the stable homotopy category, the element $E^*(f)\sigma$ of $E^*(X)$ determined by $\sigma \circ \Sigma^{-*} f : \Sigma^{-*} X \to \Sigma^{-*} Y \to E$.

Given any map $\zeta : E \to \Sigma^n E$, ζ induces a map $E^*(X) \to E^{*+n}(X)$ given by associating to $\sigma : \Sigma^{-*} X \to E$ the $(-n)$th suspension of the composite $\Sigma^{-*} X \xrightarrow{\sigma} E \to \Sigma^n E$. In other words, we have a map $E^* E \times E^* X \to E^* X$. This map is bilinear, yielding

$$E^* E \otimes E^* X \to E^* X.$$

Let $\mathcal{A} = E^* E$. $\mathcal{A}$ is a ring, and we have just seen that $E^* X$ is a module over $\mathcal{A}$.

Since $E^*(f)$ is "pre-composition with f" and ζ is "post-composition with ζ", it follows that $E^*(f)$ commutes with the action of $E^* E$, i.e. $E^*(f)$ is a morphism of modules. We therefore have

$$\pi_t F(Y, X) \to \mathrm{Hom}_{\mathcal{A}}(E^*(X), E^*(\Sigma^t Y)),$$

where the right hand side denotes homomorphisms of $\mathcal{A}$-modules. The groups Ext^s then classify extensions of $\mathcal{A}$-modules.

Warning: when working with generalized cohomology theories E, it turns out to be important to work with homology rather than cohomology. This involves trading modules for comodules, and E^*E for E_*E.