Lecture 20: Ring Spectra

4/6/15

1 Ring spectra

Let S denote the suspension spectrum of S^0.

Definition 1.1. *A* Ring spectrum *is a spectrum R equipped with an associative multiplication map $\mu : R \wedge R \to R$ with a unit $u : S \to R$. Explicitly, R is equipped with maps μ and u such that the diagrams*

$$\begin{array}{ccc} R \wedge R \wedge R & \xrightarrow{\mu \wedge 1} & R \wedge R \\ \downarrow{\scriptstyle 1 \wedge \mu} & & \downarrow{\scriptstyle \mu} \\ R \wedge R & \xrightarrow{\mu} & R \end{array}$$

and

$$\begin{array}{ccc} S \wedge R & \xrightarrow{u \wedge 1} & R \wedge R \\ \updownarrow{\scriptstyle \cong} & & \downarrow{\scriptstyle \mu} \\ R & \xrightarrow{1} & R \\ \updownarrow{\scriptstyle \cong} & & \uparrow{\scriptstyle \mu} \\ R \wedge S & \xrightarrow{1 \wedge \mu} & R \wedge R \end{array}$$

commute.

Example 1.2. *Recall that the* complex bordism spectrum *MU can be defined to be the colimit $MU = \varinjlim_n MU(n)$, where $MU(n)$ is the Thom spectrum $\Sigma^{-2n}\,\mathrm{Th}(BU(n), \zeta_n)$ of the tautological bundle over $BU(n)$ desuspended $2n$ times. The direct sum of vector bundles is classified by a map $BU(n) \times BU(m) \to BU(n+m)$, such that there is a pull-back square*

$$\begin{array}{ccc} \zeta_n \oplus \zeta_m & \longrightarrow & \zeta_{n+m} \\ \downarrow & & \downarrow \\ BU(n) \times BU(m) & \longrightarrow & BU(n+m) \end{array} .$$

Associated to this pull-back, we have a map $\mathrm{Th}(BU(n) \times BU(m), \zeta_n \oplus \zeta_m) \to \mathrm{Th}(BU(n+m), \zeta_{n+m})$. *By Lecture 2 Exercise 1.6 (3) (and using that* $S(V \times W) = SV \hat{*} SW$ *for vector bundles* V *and* W*),* $\mathrm{Th}(BU(n) \times BU(m), \zeta_n \oplus \zeta_m) \cong \mathrm{Th}(BU(n), \zeta_n) \wedge \mathrm{Th}(BU(m), \zeta_m)$. *Therefore, we have a map*

$$\mathrm{Th}(BU(n), \zeta_n) \wedge \mathrm{Th}(BU(m), \zeta_m) \to \mathrm{Th}(BU(n+m), \zeta_{n+m}),$$

whence a map

$$MU(n) \wedge MU(m) \to MU(n+m).$$

Taking colimits as $n \to \infty$ *and* $m \to \infty$, *we have a map* $\mu : MU \wedge MU$.

Note that $BU(1) = \mathbb{CP}^\infty = S^\infty/(S^1)$. *Thus the sphere bundle of* ζ_1 *is contractible. Thus* $\mathrm{Th}(BU(1), \zeta_1) = D(\zeta_1) \cong BU(1) \cong \mathbb{CP}^\infty$. *The map* $S \to MU$ *comes from the map* $S^0 \to MU(1) \cong \Sigma^{-2}\mathbb{CP}^\infty$ *given as the desuspension of* $\mathbb{CP}^1 \to \mathbb{CP}^\infty$. *(It's also possible to identify "*$MU(0) = S$*.")*

Example 1.3. *For any commutative ring* A, *the Eilenberg-MacLane spectrum* HA *is a ring spectrum. Here is an outline of an argument [Ma, Ch6.1]. For a spectrum* X, *let* $|X|$ *be the smallest* r *in* $\{r | \pi_r(X) \neq 0\}$ *if such an* r *exists. In this case,* X *can be build as a colimit* $X = \varinjlim_{n=r\to\infty} X(n)$ *where* $r = |X|$ *and* $X^{(n)}$ *is something like the* n*-skeleton. Rigorously,* $X^{(r)} = \vee S^r$ *and there are cofiber sequences* $\vee S^n \to X^{(n)} \to X^{(n+1)}$ *[Ma, Ch 3.2]. It can then be shown [Ma, Ch 3 Proposition 6] that if* Y *is a spectrum such that* $\pi_n Y = 0$ *for* $n > r$, *then*

$$[X, Y] \to \mathrm{Hom}(\pi_r(X), \pi_r(Y))$$
$$f \mapsto \pi_r(f)$$

is an isomorphism. Note that HA *therefore can be expressed as such a colimit of non-negative dimensional spheres. Since the smash product preserves colimits, the same is true of* $HA \wedge HA$. *Thus* $[HA \wedge HA, HA] \cong \mathrm{Hom}(\pi_0(HA \wedge HA), \pi_0 HA)$. *We claim that* $\pi_0(HA \wedge HA) \cong A \otimes A$. *Assuming this, we then may identify* $\mathrm{Hom}(\pi_0(HA \wedge HA), \pi_0 HA)$ *with* $\mathrm{Hom}(A \otimes A, A)$. *The multiplication on* A, *therefore gives a multiplication map* $\mu : HA \wedge HA \to HA$. *To see that* $\pi_0(HA \wedge HA) \cong A \otimes A$, *note the cofiber sequence* $HA^0 \to HA \to C$ *where* $|C| = 1$. *It follows that* $|C \wedge HA| \geq 1$. *Thus we have an exact sequence* $\pi_0(\Sigma^{-1} C \wedge HA) \to \pi_0(HA^0 \wedge HA) \to \pi_0(HA \wedge HA) \to 0$. HA^0 *is a wedge of* 0*-spheres, so* $\pi_0(HA^0 \wedge HA) = \oplus^N A$, *where* N *is the number of those* 0 *spheres. Repeat with* C, *and one has exact sequences* $\oplus^{N'} A \to \oplus^N A \to \pi_0(HA \wedge HA)$, *and* $\oplus^{N'} \mathbb{Z} \to \oplus^N \mathbb{Z} \to A \to 0$.

Definition 1.4. *Suppose* R *is a ring spectrum. A spectrum* M *is a* module *over* R *if there is a map* $\nu : R \wedge M \to M$ *in the stable homotopy category such that*

$$\begin{array}{ccc} R \wedge R \wedge M & \xrightarrow{\mu \wedge 1} & R \wedge M \\ \downarrow{\scriptstyle 1 \wedge \nu} & & \downarrow{\scriptstyle \nu} \\ R \wedge M & \xrightarrow{\nu} & M \end{array} \qquad \text{and} \qquad \begin{array}{ccc} S \wedge M & \xrightarrow{u \wedge 1} & R \wedge M \\ \updownarrow{\scriptstyle \cong} & & \downarrow{\scriptstyle \nu} \\ M & \xrightarrow{1} & M \end{array}$$

commute.

Similarly to Example 1.3, it is true that if R is an (ordinary) ring and M is an (ordinary) module over R. Then HM is an HR-module spectrum.

2 Products

Cup products in singular cohomology generalize to products in generalized cohomology. Recall associated to a spectrum E, there is a generalized cohomology theory E^n for $n \in \mathbb{Z}$, where E^n takes a spectrum X to the abelian group $\pi_{-n}F(X,E)$.

Let E and F be spectra. The *external product in cohomology* is a map

$$E^p(X) \otimes F^q(Y) \to (E \wedge F)^{p+q}(X \wedge Y)$$

defined as follows. An element of $\sigma \in E^p(X)$ is an element of $[S^{-p}, F(X,E)] \cong [S^{-p} \wedge X, E]$, so we have a morphism $\sigma : S^{-p} \wedge X \to E$ in the stable homotopy category. Similarly, given $\tau \in F^q(Y)$, we have a morphism $\tau : S^{-q} \wedge Y \to F$. We form $\sigma \wedge \tau : S^{-p} \wedge X \wedge S^{-q} \wedge Y \to E \wedge F$. Permuting the factors gives a map

$$S^{-(p+q)} \wedge X \wedge Y \to E \wedge F,$$

and therefore an element of $[S^{-(p+q)}, F(X \wedge Y, E \wedge F)] \cong (E \wedge F)^{p+q}(X \wedge Y)$. Once we have an external product in cohomology, we obtain an external product for a based CW-complex, by taking the suspension spectrum. Similarly, we have an external product relative groups.

Exercise 2.1. *Recall, that for an inclusion $A \subset X$ of CW-complexes, the generalized relative cohomology is defined by $E^n(X,A) = E^n(\Sigma^\infty(X/A))$. If (X,A) and (Y,B) are CW-pairs, show that there is a product*

$$E^p(X,A) \otimes F^q(Y,B) \to (E \wedge F)^{p+q}(X \times Y, A \times Y \cup X \times B).$$

Taking $F = E$, we have $E^p(X) \otimes E^q(X) \to (E \wedge E)^{p+q}(X \wedge X)$. If E is a ring spectrum, then we can compose with the map $E \wedge E \to E$ to obtain a product $E^p(X) \otimes E^q(X) \to E^{p+q}(X \wedge X)$. If X is a CW-complex, then there is a diagonal map $X \to X \times X$. Adding a disjoint base-point, we have $X_+ \to (X \times X)_+ \cong (X_+) \wedge (X_+)$. This gives rise to a diagonal map $\Sigma^\infty X_+ \to (\Sigma^\infty X_+) \wedge (\Sigma^\infty X_+)$. For a CW-complex, use the abbreviation $E^n(X) = E^n(\Sigma^\infty X_+)$. (With a base point, we could also define $\tilde{E}^n(X) = E^n(\Sigma^\infty X)$, and this convention is consistent with the notion of reduced and unreduced cohomology.) In total, we get a product

$$E^p(X) \otimes E^q(X) \to E^{p+q}X.$$

Example 2.2. *For $H\mathbb{Z}$, this is the usual cup product.*

Exercise 2.3. *Define the cap product similarly. (If you get stuck, see [A, III 9 slant product])*

References

[A] J.F. Adams, *Stable Homotopy and Generalized Homology* Chicago Lectures in Mathematics, The University of Chicago Press, 1974.

[Ma] H.R. Margolis *Spectra and the Steenrod Algebra*