# Lecture 1: stable homotopy theory

1/5/14

## 1 Stability

A phenomenon in homotopy theory is *stable* if it occurs in all sufficiently large dimensions in essentially the same way. Otherwise, it is *unstable*. The meaning of sufficiently large usually depends on the connectivity of the spaces involved. There is some confusing terminology associated with connectivity, so let's recall the following definitions. Write $[X, Y]$ for the set of base-point preserving homotopy classes of maps $X \to Y$, where $X$ and $Y$ are based CW-complexes.

**Definition 1.1.** *$X$ is $n$-connected if for all base points and all $k \leq n$ the homotopy group $\pi_k(X) := [S^k, X]$ is trivial.*

For example, if $X$ is a CW-complex of dimension $\leq n$ and $Y$ is $n$-connected, then any map $X \to Y$ is null-homotopic. The wedge product $X \vee Y$ of spaces $X$ and $Y$ is the coproduct in based spaces and is defined by taking the disjoint union and identifying the base points. The smash product $X \wedge Y$ of $X$ and $Y$ is defined to be

$$X \wedge Y = X \times Y/(X \vee Y),$$

but it isn't the categorical product in based spaced; that's still $X \times Y$. The (reduced) suspension $\Sigma X$ of $X$ is defined $\Sigma X = S^1 \wedge X$. The suspension of a map $f : X \to Y$ is $1 \wedge f : S^1 \wedge X \to S^1 \wedge Y$, defining a function $\Sigma$ (or $E$)

$$\Sigma : [X, Y] \to [\Sigma X, \Sigma Y].$$

**Theorem 1.2.** *If $Y$ is $n-1$ connected, then $E$ is surjective if* $\dim X \leq 2n-1$ *and bijective if* $\dim X < 2n-1$.

When suspension induces an isomorphism, maps are called stable.

**Example 1.3.** $\Sigma : [S^3, S^2] \cong \mathbb{Z}\eta \to [S^4, S^3] \cong \mathbb{Z}/2$ *is surjective. The map* $\eta : S^3 = \{(z_1, z_2) \in \mathbb{C}^2 : |z_1|^2 + |z_2|^2 = 1\} \to S^2 = \mathbb{CP}^1$ *defined* $\eta(z_1, z_2) = z_1/z_2$ *is the* Hopf map. $\Sigma : [S^4, S^3] \cong \mathbb{Z}/2 \to [S^5, S^4] \cong \mathbb{Z}/2$ *is an isomorphism. (You can use this example to remember the $n-1$, $2n-1$, $<$ etc.)*

It follows from the Hurewicz theorem (see for example [H, 4.2 p. 366]) that the suspension of an $n$-connected space, is $n+1$-connected. Thus when $\dim X$ is finite, $2n-1-\dim X$ increases by one under stabilization $\Sigma$. So, all maps become stable after sufficiently many suspensions. Stable homotopy classes of maps are easier to compute because they form generalized (co)homology theories. We'll define and discuss these in a few lectures. Today we'll discuss some applications of stable homotopy theory.

# 2 A few applications

**Question 2.1.** *What is the maximum number of linearly independent vector fields on $S^{n-1}$?*

This question was solved by Adams: $S^{n-1}$ has $\rho(n)-1$ linearly independence vector fields, but not $\rho(n)$, where $\rho(n)$ is the $n$th Radon-Hurwitz number. $\rho(n)$ is computed by expressing $n$ as $n = (2a+1)2^b$ and $b = c + 4d$ with $0 \leq c < 4$, then setting $\rho(n) = 2^c + 8d$.

**Problem 2.2.** *Classify compact oriented smooth $n$-manifolds up to a certain equivalence relation called cobordism.*

Denote the resulting group by $\Omega_n$. It is not clear that this problem is even in homotopy theory, but it is, and moreover it turns out to be stable: René Thom introduced the Thom complex $MSO(m)$ and gave an isomorphism

$$\Omega_n \cong \pi_{m+n}(MSO(m))$$

for $m > n+1$.

**Question 2.3.** *For which $n$ is $S^{n-1}$ an $H$-space?*

For example, $S^3$ is an $H$-space, but $S^5$ is not. The answer is "yes" if and only if $n = 1, 2, 4, 8$, and was also solved by Adams. This problem is unstable, but it can be solved with stable homotopy theory. There exist natural transformations

$$\mathrm{Sq}^n : \mathrm{H}^m(X, \mathbb{Z}/2) \to \mathrm{H}^{m+n}(X, \mathbb{Z}/2)$$

called *Steenrod operations* which are examples of *stable cohomology operations*. The question can be reduced to the following problem. Suppose that $m \geq n$. Is there a CW complex $X = S^m \cup e^{m+n}$ such that $\mathrm{Sq}^n$ is non-zero? This question is also equivalent to the Hopf invariant 1 problem, which asks for which $n$ does there exist a map $f : S^{2n-1} \to S^n$ with Hopf invariant one? (Answer:

n=2,4,8) .The Hopf invariant $H(f)$ of $f$ is defined as follows. Form the complex $X = S^n \cup e^{2n}$ where the attaching map is $f$. Equivalently,

$$S^{2n-1} \xrightarrow{f} S^n \to X$$

is a cofiber sequence. Then $\mathrm{H}^*(X) = \mathbb{Z}$ or 0 with $\mathbb{Z}$ exactly when $* = 0, n, 2n$. Let $x$ be a generator in dimension $n$ and $y$ be a generator in dimension $2n$. Then $x^2 = H(f)y$. There are pretty pictures of this invariant. For example, for a map $S^3 \to S^2$, $H(f)$ can be computed as the linking number of the two 1-manifolds given by inverse images of two chosen points in $S^2$. See [S]. This question can also be phrased in terms of the $s = 1$ line of the Adams spectral sequence for the stable homotopy groups of the sphere.

**Question 2.4.** *For which $n$, does there exist a smooth, stably framed $n$-manifold with Kervaire invariant one?*

Although this question dates from the 1960s, it wasn't answered until 2009. Suppose that $n$ is congruent to 2 mod 4 and that $M$ is a smooth stably framed $n$-manifold. Then there is a quadratic refinement $q : \mathrm{H}^{n/2}(M, \mathbb{Z}/2) \to \mathbb{Z}/2$ of the intersection pairing, i.e.,

$$q(x + y) = q(x) + q(y) + \langle x \cup y, [M] \rangle,$$

coming from a calculation that the $n$th stable homotopy group of $K(\mathbb{Z}/2, n/2)$ is $\mathbb{Z}/2$. The Arf invariant of such a quadratic form is 0 if and only if $q$ takes the value 0 more often than it takes the value 1. Otherwise the Arf invariant is 1. The Kervaire invariant of $M$ is the Arf invariant of $q$. The answer to the question is "yes" if and only if $n = 2, 6, 14, 30, 62$ and possibly 126. It is due to Hill, Hopkins, and Ravanel. The $n = 126$ case is still open. This question can also be phrased in terms of the $s = 2$ line of the Adams spectral sequence for the stable homotopy groups of the sphere. See [M].

# References


[A] J.F. Adams, *Stable Homotopy and Generalized Homology* Chicago Lectures in Mathematics, The University of Chicago Press, 1974.

[H] Allen Hatcher, *Algebraic Topology.*

[M] Haynes Miller, *Kervaire Invariant One (after M.A. Hill, M.J. Hopkins, and D.C. Ravenel*, Séminarie BOURBAKI, 63éme année, 2010-2011, n. 1029.

[S] N.E. Steenrod, *Cohomology Invariants of Mappings*, Annals of Math, 1949.