

# Lecture 19: Abstract homotopy theory

3/27/15

In the last lecture we constructed various categories of spectra. We have been using homotopy theory of spectra, so we need a way to add homotopy theory to these categories. In this lecture, we discuss two standard ways to bring homotopy theory into category theory.

## 1 Model categories

Before defining a model category, we need three definitions. A map  $A \rightarrow B$  is said to be a *retract* of a map  $C \rightarrow D$  if there exists a commutative diagram

$$\begin{array}{ccccc} A & \longrightarrow & C & \longrightarrow & A \\ \downarrow & & \downarrow & & \downarrow \\ B & \longrightarrow & D & \longrightarrow & B \end{array}$$

such that the composition  $A \rightarrow A$  and  $B \rightarrow B$  on the top and bottom are both the identity.

Secondly, let  $i : Y \rightarrow X$  and  $p : E \rightarrow B$  be two maps in a category  $\mathcal{C}$ . The morphism  $i$  is said to have the *left lifting property* with respect to  $p$  if for all commutative diagrams

$$\begin{array}{ccc} Y & \xrightarrow{f} & E \\ \downarrow i & & \downarrow p \\ X & \xrightarrow{g} & B \end{array}$$

there exists a morphism  $h : X \rightarrow E$  such that  $f = hi$  and  $g = ph$ . In the same situation, say that  $p$  has the *right lifting property* with respect to  $i$ .

For example if  $i$  is the inclusion  $i : X \rightarrow X \times [0, 1]$  and  $p$  is a covering space, then  $i$  has the left lifting property with respect to  $p$ .

Thirdly, a *functorial factorization*  $(\alpha, \beta)$  is pair of functors from the category of the maps in  $\mathcal{C}$  to itself (e.g.  $\alpha$  takes a map  $f$  in  $\mathcal{C}$  and creates another map in  $\mathcal{C}$  called  $\alpha(f)$ ) such that  $f = \beta(f)\alpha(f)$ .

**Definition 1.1.** ([Ho, Definition 1.1.3]) *A model category is a category  $\mathcal{C}$  which admits all small limits and colimits (the adjective “small” means that the category over which the limit or colimit is taken has a set of objects) together with three subcategories called weak equivalences, cofibrations, and fibrations, and two functorial factorizations  $(\alpha, \beta)$  and  $(\gamma, \delta)$  such that:*

1. (2-out-of-3) *If  $f$  and  $g$  are morphisms of  $\mathcal{C}$  such that  $gf$  is defined and two of  $f, g$  and  $gf$  are weak equivalences, then so is the third.*
2. (Retracts) *If  $f$  and  $g$  are morphisms of  $\mathcal{C}$  such that  $f$  is a retract of  $g$  and  $g$  is a weak equivalence, cofibration, or fibration (respectively), then so is  $f$ .*
3. (Lifting) *Define a map to be a trivial cofibration if it is both a cofibration and a weak equivalence. Similarly, define a map to be a trivial fibration if it is both a fibration and a weak equivalence. Then trivial cofibrations have the left lifting property with respect to fibrations, and cofibrations have the left lifting property with respect to the trivial fibrations.*
4. (Factorization) *For any morphism  $f$ ,  $\alpha(f)$  is a cofibration,  $\beta(f)$  is a trivial fibration,  $\gamma(f)$  is a trivial cofibration, and  $\delta(f)$  is a fibration.*

For example, topological spaces can be made into a model category where the weak equivalences are  $\pi_*$ -isomorphisms and the cofibrations are retracts of relative cell complexes [Ho, Theorem 2.4.19]. The mapping cylinder construction gives the functorial factorization  $(\alpha, \beta)$  and the path space construction gives the functorial factorization  $(\gamma, \delta)$  in topological spaces.

In fact, once the cofibrations and weak equivalences of a model category are known, the fibrations etc. are determined. To see this:

**Lemma 1.2.** (Retract argument)

- *Suppose  $f = pi$  and  $f$  has the left lifting property with respect to  $p$ . Then  $f$  is a retract of  $i$ .*
- *Suppose  $f = pi$  and  $f$  has the right lifting property with respect to  $i$ . Then  $f$  is a retract of  $p$ .*

*Proof.* Consider the diagram

$$\begin{array}{ccc} A & \xrightarrow{i} & C \\ \downarrow f & & \downarrow p \\ B & \xrightarrow{1} & B \end{array}$$

We have a lift  $h : B \rightarrow C$  such that adding  $h$  to the diagram keeps the diagram commutative. Then  $f$  is a retract of  $i$  by the diagram

$$\begin{array}{ccccc} A & \xrightarrow{1} & A & \xrightarrow{1} & A \\ \downarrow f & & \downarrow i & & \downarrow f \\ B & \xrightarrow{h} & C & \xrightarrow{p} & B \end{array} .$$

For the second claim, the diagram

$$\begin{array}{ccc} A & \xrightarrow{1} & A \\ \downarrow i & & \downarrow f \\ C & \xrightarrow{p} & B \end{array} ,$$

admits a lift  $h : C \rightarrow A$ , showing that  $f$  is a retract of  $p$  by the diagram

$$\begin{array}{ccccc} A & \xrightarrow{i} & C & \xrightarrow{h} & A \\ \downarrow f & & \downarrow p & & \downarrow f \\ B & \xrightarrow{1} & B & \xrightarrow{1} & B \end{array} .$$

□

**Exercise 1.3.** Use the Retract argument to show the following:

- A map is a fibration (respectively a trivial fibration) if and only if it has the right lifting property with respect to all trivial cofibrations (respectively fibrations).
- A map is a cofibration (respectively a trivial cofibration) if and only if it has the left lifting property with respect to all trivial fibrations (respectively fibrations).

Thus, if the weak equivalences are given, then the fibrations determine the cofibrations, and the cofibrations determine the fibrations.

Given a model category  $\mathcal{C}$ , the associated *homotopy category*  $\text{ho}\mathcal{C}$  is the category whose objects are the same as the objects of  $\mathcal{C}$  and whose morphisms are finite strings of composable arrows where each arrow is either a morphism of  $\mathcal{C}$  or the reverse of a weak equivalence. Composition is by concatenation. (We should really be concerned about whether the morphisms just defined are really a set, but this turns out not to be important.)

The homotopy category also has an equivalent description that looks more like taking homotopy classes of maps. Namely, suppose we wish to define the homotopy equivalence relation on maps  $X \rightarrow Y$ . If we had an interval object  $[0, 1]$  with the inclusion of two endpoints  $* \coprod * \xrightarrow{\{0,1\}} [0, 1]$  we could say that two maps  $f, g : X \rightarrow Y$  were homotopic if there was a map  $X \times [0, 1] \rightarrow Y$  whose pull-back to  $X \times (* \coprod *) \cong X \coprod X \rightarrow Y$  is  $f \coprod g$ . Now, we can make an appropriate substitute for  $X \times [0, 1]$  by taking the map  $X \coprod X \rightarrow X$  and factoring it as  $\beta\alpha$  where  $\alpha : X \coprod X \rightarrow Z$  is a cofibration and  $\beta : Z \rightarrow X$  is a trivial fibration. Then  $Z$  is a replacement for  $X \times [0, 1]$ . Now define  $f, g : X \rightarrow Y$  to be homotopic if there is a map  $Z \rightarrow Y$  which pulls back to  $f \coprod g : X \coprod X \rightarrow Y$ . This statement only works well when we restrict to  $X$  and  $Y$  such that  $\emptyset \rightarrow X$  is a cofibration and  $Y \rightarrow *$  is a fibration. ( $\emptyset$  exists because it is the colimit over the empty set and  $*$  exists because it is the limit over the empty set.) It turns out that there is an equivalence of categories

$$\{X | \emptyset \rightarrow X \text{ is a cofibration, } X \rightarrow * \text{ is a fibration}\} / \sim \cong \text{ho}\mathcal{C},$$

where on the left hand side, the equivalence relation is homotopy [Ho, Theorem 1.2.10].

When does a functor between model categories preserve enough of the structure to be considered a morphism of model categories? The name for such functors are *Quillen functors*, and they are required to come in adjoint pairs.

Let  $\mathcal{C}$  and  $\mathcal{D}$  be model categories.

**Definition 1.4.**     •  $P : \mathcal{C} \rightarrow \mathcal{D}$  is a left Quillen functor if it is a left adjoint and preserves cofibrations and trivial cofibrations.

•  $U : \mathcal{D} \rightarrow \mathcal{C}$  is a right Quillen functor if it is a right adjoint and preserves fibrations and trivial fibrations.

•  $(P, U)$  is a Quillen adjunction if  $P : \mathcal{C} \rightarrow \mathcal{D}$  is left adjoint to  $U : \mathcal{D} \rightarrow \mathcal{C}$ , and  $P$  is a left Quillen functor.

The definition of a Quillen adjunction looks asymmetrical, but it isn't:

**Lemma 1.5.** *If  $P : \mathcal{C} \rightarrow \mathcal{D}$  is left adjoint to  $U : \mathcal{D} \rightarrow \mathcal{C}$ , then  $(P, U)$  is a Quillen adjunction if and only if  $P$  is a right Quillen functor.*

*Proof.* Since  $(P, U)$  are adjoint (which gives natural bijections such as  $\text{Hom}_{\mathcal{D}}(PX, E) \cong \text{Hom}_{\mathcal{C}}(X, UE)$ ), it follows that  $Pi$  has the left lifting property with respect to  $p$  if and only if  $i$  has the left lifting property with respect to  $Up$ . By Exercices 1.3, it follows that  $P$  preserves (trivial) cofibrations if and only if  $U$  preserves (trivial) fibrations.  $\square$

**Example 1.6.** Let  $\mathbf{K}$  denote the category of compactly generated, weak Hausdorff (i.e. continuous images of any compact Hausdorff space is closed) topological spaces. Let  $\Delta$  be the category whose objects are  $[n] = \{1, 2, \dots, n\}$  and whose morphisms are the morphisms of sets which preserve  $\leq$ . A simplicial set is a functor  $X : \Delta \rightarrow \mathbf{Set}$ . Let  $\mathbf{sSet}$  denote the category of simplicial sets. Then the geometric realization and singular complex define a Quillen equivalence  $\mathbf{sSet} \rightarrow \mathbf{K}$ .

A left Quillen functor  $P : \mathcal{C} \rightarrow \mathcal{D}$  determines a *derived functor*  $LP : \text{ho } \mathcal{C} \rightarrow \text{ho } \mathcal{D}$  as follows. For any object  $X$  in  $\mathcal{C}$  the morphism  $\emptyset \rightarrow X$  may be factored as a cofibration  $\emptyset \rightarrow QX$  followed by a trivial fibration  $QX \rightarrow X$ . By the definition of a model category,  $Q$  is a functor. The left derived functor  $LP$  is defined  $LP(X) = P(QX)$ . There is a similar story for a right Quillen functor. A Quillen adjunction  $(P, U)$  such that the associated adjoint functors between  $\text{ho } \mathcal{C}$  and  $\text{ho } \mathcal{D}$  determines an equivalence of categories is a *Quillen equivalence* [Ho, Proposition 1.3.13]. (For a more useful definition of Quillen equivalence, see [Ho, Definition 1.3.12].)

It is possible to put model structures on the categories  $\mathbf{NS}_S$  (prespectra),  $\mathbf{\Sigma S}_S$  (symmetric spectra),  $\mathbf{JS}_S$  (orthogonal spectra) so that all of the resulting homotopy categories are the stable homotopy category from Lecture 3. This is done in [MMSS]. I think it would be too technical for us to go through this process, but the model structure on  $\mathbf{NS}_S$  is compatible with what we called *cofiber* sequences before. In  $\mathbf{NS}_S$  and  $\mathbf{JS}_S$ , the weak equivalences are precisely the  $\pi_*$  isomorphisms, meaning the maps  $f : X \rightarrow Y$  such that  $\pi_*(X) \rightarrow \pi_*(Y)$  is an isomorphism for all  $*$ . (Define  $\pi_*X$  to be the maps in the homotopy category from  $S^*$  wedge the sphere spectrum to  $X$ .) Moreover, these model structures are all Quillen equivalent under the prolongation functor  $\mathbb{P}$  and the underlying spectrum functor  $\mathbb{U}$  from Lecture 18. See [MMSS, Section 10, Theorem 10.4, Corollary 10.5]. This implies that the homotopy categories are equivalent. This gives us a smash product and function spectrum as we wished.

Recall that in [A], the morphisms in the stable homotopy category, were defined in stages. First, Adams defined maps of spectra, then maps from cofinal subspectra (we called these pmaps), and then modded out by homotopies of maps on cofinal subspectra. Note that the inclusion of a cofinal subspectrum is a  $\pi_*$ -isomorphism, so when we invert  $\pi_*$ -isomorphisms, a map out of a cofinal subspectrum of  $X$  also gives us a map out of  $X$ . Likewise, the definition of homotopic pmaps in [A] is compatible with the notion of a cylinder object

above, so we have a well-defined functor from the stable homotopy category of Lecture 3. This map is an equivalence. In other words, Adams identified a nice short list of modifications to the category of spectra that produced the correct stable homotopy category.

## 2 Infinity categories

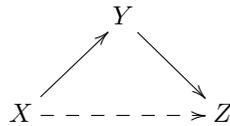
Recall from Example 1.6 that  $\mathbf{sSet}$  denotes the category of simplicial sets. The  $n$ -simplex  $\Delta^n$  is  $\Delta^n = \text{Hom}(-, [n])$  the simplicial set represented by  $[n]$ . Let  $\Lambda_i^n$  denote simplicial subset of  $\Delta^n$  obtained by removing the interior and the face opposite the vertex  $i$ .

**Definition 2.1.** *An infinity category is a simplicial set  $C$  with the property that for every map  $f : \Lambda_i^n \rightarrow C$  with  $0 < i < n$ , the map  $f$  extends to a map  $\Delta^n \rightarrow C$ .*

They are generalizations of categories by the following construction. For a category  $\mathcal{C}$ , the *Nerve* of  $\mathcal{C}$  is the simplicial set whose  $n$ -simplices are given by strings of composable morphisms  $C_0 \rightarrow C_1 \rightarrow \dots \rightarrow C_n$ . Note that  $\mathcal{C}$  can be reconstructed from  $N(\mathcal{C})$ .

**Example 2.2.** *For any category  $\mathcal{C}$ , the nerve  $N(\mathcal{C})$  is an  $\infty$ -category.*

Composition in an  $\infty$ -category comes from the horn-filling condition for the horn  $\Lambda_1^2$ . Namely, two composable morphisms  $f$  and  $g$  give a map  $\Lambda_1^2 \rightarrow C$ . By hypothesis, we can extend this map to a map  $\Delta^2 \rightarrow C$ .



The 1-face of this extension defines a composition. This composition is not necessarily unique, but it is unique up to a suitable definition of homotopy.

**Exercise 2.3.** *Let  $X$  be a topological space. The singular complex  $\text{Sing } X$  of  $X$  is defined to be the simplicial set  $\text{Sing } X[n] = \mathcal{T}(\Delta^n, X)$  whose  $n$ -simplices are all the continuous maps from  $\Delta^n$  to  $X$ . Show that  $\text{Sing } X$  is an  $\infty$ -category.*

The notion of an  $\infty$ -category is close to the notion of a topological category. Namely, let  $\mathcal{C}$  be a topological category. There is an associated  $\infty$ -category  $N^t(\mathcal{C})$  called the *homotopy coherent nerve*. The 0-simplices  $N^t(\mathcal{C})_0$  of  $N^t(\mathcal{C})$

are the objects of  $\mathcal{C}$ . The elements of  $N^t(\mathcal{C})_1$  are morphisms  $X \rightarrow Y$  in  $\mathcal{C}$ . The elements of  $N^t(\mathcal{C})_2$  are not-necessarily commuting diagrams

$$\begin{array}{ccc} & Y & \\ g \nearrow & & \searrow f \\ X & \xrightarrow{h} & Z \end{array}$$

together with a path in  $\mathcal{C}(X, Y)$  from  $h$  to  $g \circ f$ . The data determining the elements of  $N^t(\mathcal{C})_n$  are defined similarly. Conversely, any infinity category is equivalent to  $N^t(\mathcal{C})$  for some topological category  $\mathcal{C}$ , and this  $\mathcal{C}$  is unique up to a suitable notion of equivalence.

**Example 2.4.** *From the topological category of topological spaces, we then also have an  $\infty$ -category of topological spaces. The category  $\mathbf{sSet}$  is also a topological category. Namely, given simplicial sets  $X$  and  $Y$ , define the simplicial set  $F(X, Y)$  by  $F(X, Y)_n = \mathbf{sSet}(X \times \Delta^n, Y)$ . So, we also have an  $\infty$ -category of simplicial sets.*

Given an infinity category  $C$ , there is an associated *homotopy category*  $\mathrm{ho} C$ . If  $\mathcal{C}$  is the associated topological category, this homotopy category is the category with the same objects, and whose morphisms from  $X$  to  $Y$  are  $\pi_0 \mathcal{C}(X, Y)$ . Without resorting to the topological category,  $\mathrm{ho} C$  can be described: the objects of  $\mathrm{ho} C$  are the vertices of  $C$ . For every vertex, the degenerate one simplex on that vertex is the identity. (i.e., the map  $[1] \rightarrow [0]$  determines a map  $s_0 : C_0 \rightarrow C_1$ , and  $s_0 c$  is the identity at  $c$  for all  $c \in C_0$ .) For every edge  $\phi : \Delta^1 \rightarrow C$ , there is a morphism  $\bar{\phi}$  from  $\phi(0)$  to  $\phi(1)$ . For every  $\sigma : \Delta^2 \rightarrow C$  there is a relation  $\overline{d_0 \sigma} \circ \overline{d_2 \sigma} = \overline{d_1 \sigma}$ , where  $d_i$  denotes the side of  $\Delta^2$  opposite vertex  $i$ . See [LuHTT, 1.2.3].

There are good notions of limits and colimits in an infinity category [LuHTT, 1.2.13]. To appreciate this fact, consider first the homotopy category. In the homotopy category, limits and colimits often do not exist. For example, it is tempting to say that the mapping cone  $C(f)$  of a map  $f : X \rightarrow Y$  should be the colimit in  $\mathrm{ho} \mathcal{T}$  of the diagram

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \downarrow & & \\ * & & \end{array}$$

After all, given a map  $Y \rightarrow Z$ , and a nullhomotopy of its restriction to  $X$ , we can define a map  $C(f) \rightarrow Z$ . However, this map is not necessarily unique up to homotopy, i.e. we have not determined a map in  $\mathrm{ho} \mathcal{T}$  from  $C(f)$  to  $Z$ . Keeping track of the homotopies fixes this problem.

We can formally invert suspension in the setting of infinity categories and obtain an infinity category. There is an  $\infty$ -category  $\mathbf{Cat}_\infty$  whose objects are (small) infinity categories. Let  $\mathbf{sSet}$  denote the  $\infty$ -category of simplicial sets. Suspension  $\Sigma X = S^1 \wedge X$  gives a functor  $\Sigma : \mathbf{sSet} \rightarrow \mathbf{sSet}$ . We can define the  $\infty$ -category of spectra to be

$$\operatorname{colimit}(\dots \rightarrow \mathbf{sSet} \xrightarrow{\Sigma} \mathbf{sSet} \xrightarrow{\Sigma} \mathbf{sSet} \xrightarrow{\Sigma} \mathbf{sSet} \rightarrow \dots),$$

where this colimit is taken in  $\mathbf{Cat}_\infty$ . Passing to the homotopy category, we get the stable homotopy category of lecture 3. See [Ro13, Prop 4.21]. Moreover, the smash product on  $\mathbf{sSet}$  can be extended to spectra [Ro13, Corollaries 4.24, 4.25].

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