Lecture 18: Smash product

3/23/15

Recall that given two topological spaces X and Y with base points, we have the smash product $X \wedge Y = X \times Y/(X \vee Y)$ and a space of functions preserving base points $F(X,Y)$ such that $F(X, F(Y,Z)) \cong F(X \wedge Y, Z)$. We wish to have a smash product and an internal function object for spectra. With the category of spectra as defined in Lecture 2, it is not possible to have an associative and commutative smash product. So we expand our notion of what a spectrum is. The notions of spectra here will all give rise to the stable homotopy category of Lecture 3, and we will address this point in Lecture 19. Here we introduce categories of spectra with smash product and internal function objects.

1 Categories of $\mathcal{D}$-spaces

We follow [MMSS]. Topological categories are categories where the morphisms between objects form a topological space and composition is continuous. A functor $X : \mathcal{D} \to \mathcal{C}$ between topological categories is *continuous* if for any objects d, e in $\mathcal{D}$ the induced map $\mathcal{D}(d,e) \to \mathcal{C}(X(d), X(e))$ is continuous. Let $\mathcal{T}$ be the topological category of compactly generated topological spaces (X is compactly generated means that a subspace is open if and only if its inverse image under any continuous functions from a compact Hausdorff spaces is open) equipped with a base point.

Definition 1.1. *Let $\mathcal{D}\mathcal{T}$ denote the category of continuous functors $\mathcal{D} \to \mathcal{T}$ and natural transformations. Objects of this category are called $\mathcal{D}$-spaces.*

A *monoidal category* is a category $\mathcal{D}$ with an operation $\square : \mathcal{D} \times \mathcal{D} \to \mathcal{D}$ and a unit object u such that $\square$ is associative with unit u up to natural isomorphism, which is required to be coherent in the sense that reasonable diagrams constructed with the natural isomorphisms also commute. See the references in [MMSS, §20 p. 58]. If $\square$ is commutative up to coherent natural isomorphism, $\mathcal{D}$ is a symmetric monoidal category.

Example 1.2. *Let $\mathbb{N}$ be the category whose objects are non-negative integers, and with only identity morphisms. The symmetric monoidal structure is addition, and* 0 *is the unit.*

Example 1.3. *Let Σ be the category whose objects are finite sets $\{1, 2, \ldots, n\}$ such that $n \geq 0$, with morphisms given by bijections.The symmetric monoidal structure is*

$$\{1, 2, \ldots, n\}\Box\{1, 2, \ldots, m\} = \{1, 2, \ldots, n+m\}$$

with block sum of permutations. $n = 0$ is the unit.

Example 1.4. *Let $\mathcal{I}$ be the category of finite dimensional real inner product spaces and linear isometries. There are no maps $V \to W$ unless V and W have the same dimension n, and in this case, the space of morphisms is homeomorphic to $O(n)$. The symmetric monoidal structure is orthogonal direct sum and the* 0 *dimensional vector space is the unit.*

For a symmetric monoidal topological category $\mathcal{D}$, we can equip $\mathcal{D}\mathcal{T}$ with a commutative, associative smash product with unit and an internal Hom functor F such that

$$F(X, F(Y, Z)) \cong F(X \wedge Y, Z) \tag{1}$$

for all X, Y, Z in $\mathcal{D}\mathcal{T}$. To do this, we first define the *external smash product* which takes two $\mathcal{D}$-spaces $X, Y : \mathcal{D} \to T$ and forms

$$X \overline{\wedge} Y = \wedge \circ (X \times Y) : \mathcal{D} \times \mathcal{D} \to \mathcal{T}.$$

We then define the smash product (or *internal smash product*) by

$$(X \wedge Y)(d) = \operatorname*{colim}_{e\Box f \to d} X(e) \wedge Y(f).$$

Here, the colimit is taken over the category whose objects are pairs $(e, f) \in \mathcal{D}^2$ and maps $\alpha : e\Box f \to d$, and a morphism from $\{(e, f), \alpha\}$ to $\{(e', f'), \alpha'\}$ is a pair of maps $\phi : e \to e'$ and $\varphi : f \to f'$ such that $\alpha' \circ (\phi \oplus \varphi) = \alpha$. (We should really restrict to circumstances where this category can be assumed to be small. Also note that $X \wedge Y$ is a $\mathcal{D}$-space by functoriality of colimits.) For a concrete description of $X \wedge Y(d)$, recall that the coequalizer of a diagram $A \rightrightarrows B$ where the two morphisms are called p_1 and p_2 is $C = B/(p_1(a) \sim p_2(a))$, and we write

$$A \rightrightarrows B \to C$$

as a coequalizer sequence. Then

$$\vee_{(e',f')\to(e,f)}\mathcal{D}(e\Box f, d)\wedge(X(e')\wedge Y(f')) \rightrightarrows \vee_{(e,f)}\mathcal{D}(e\Box f, d)\wedge(X(e)\wedge Y(f)) \to (X\wedge Y)(d)$$

is a coequalizer sequence. This is an example of a left Kan extension.

Exercise 1.5. *Show that* $(X \wedge Y)(n) = \vee_{p=0}^{n} (\Sigma_n)_+ \wedge_{\Sigma_p \times \Sigma_{n-p}} X(p) \wedge Y(n-p)$ *for* X *and* Y *two* Σ*-spaces.*

Let X and Y be $\mathcal{D}$-spaces. Define the *function* $\mathcal{D}$-space $F(X, Y)$ by

$$F(X,Y)(d) = \mathcal{D}\mathcal{T}(X, Y(d\square -)).$$

We can now prove (1): Note the canonical projections $\mathcal{D}\mathcal{T}(X, F(Y,Z)) \to \mathcal{T}(X(d), \mathcal{D}\mathcal{T}(Y, Z(d\square -)))$, which determine $\mathcal{D}\mathcal{T}(X, F(Y,Z))$ by the exact sequence

$$\mathcal{D}\mathcal{T}(X, F(Y,Z)) \to \prod_{d} \mathcal{T}(X(d), \mathcal{D}\mathcal{T}(Y, Z(d\square -))) \to \prod_{d\to d'} \mathcal{T}(X(d), \mathcal{D}\mathcal{T}(Y, Z(d'\square -))).$$

We furthermore have projections $\mathcal{D}\mathcal{T}(Y, Z(d\square -))) \to \mathcal{T}(Y(e), Z(d\square e))$ with an analogous exact sequence determining $\mathcal{D}\mathcal{T}(Y, Z(d\square -)))$. Combining with the natural homeomorphisms $\mathcal{T}(X(d), \mathcal{T}(Y(e), Z(d\square e))) \cong \mathcal{T}(X(d) \wedge Y(e), Z(d\square e))$, we have

$$\mathcal{D}\mathcal{T}(X, F(Y,Z)) \cong (\mathcal{D} \times \mathcal{D})\mathcal{T}(X \overline{\wedge} Y, Z \circ \square). \tag{2}$$

We now claim that

$$(\mathcal{D} \times \mathcal{D})\mathcal{T}(X \overline{\wedge} Y, Z \circ \square) \cong \mathcal{D}\mathcal{T}(X \wedge Y, Z). \tag{3}$$

To see this, we first expand the colimit $X \wedge Y$ as a coequalizer

$$\vee_{(e',f')\to(e,f)} \mathcal{D}(e\square f, d) \wedge (X(e') \wedge Y(f')) \rightrightarrows \vee_{(e,f)} \mathcal{D}(e\square f, d) \wedge (X(e) \wedge Y(f)) \to (X \wedge Y)(d).$$

Given a map $X \overline{\wedge} Y \to Z \circ \square$, we obtain a map $\vee_{(e,f)} \mathcal{D}(e\square f, d) \wedge (X(e) \wedge Y(f)) \to \vee_{(e,f)} \mathcal{D}(e\square f, d) \wedge (Z(e\square f)) \to Z(d)$. This map determines a map $X \wedge Y \to Z$, by the coequalizer sequence. Conversely, given a map $X \wedge Y \to Z$, we obtain a map $X \overline{\wedge} Y \to Z \circ \square$ by letting the map $X(e) \wedge Y(f) \to Z(e\square f)$ come from the identity point of $\vee_{(e,f)} \mathcal{D}(e\square f, e\square f) X(e) \wedge Y(f)$. These are inverse homeomorphisms, showing (3). Combining (2) and (3) shows (1).

Exercise 1.6. *Define* u^* *to be the* $\mathcal{D}$*-space* $e \mapsto \mathcal{D}(u, e)$*. Show that* u^* *is a unit for the operation* $\wedge$ *on* $\mathcal{D}$*-spaces.*

It follows that $\mathcal{D}\mathcal{T}$ is a symmetric monoidal topological category, with operation $\wedge$ and unit u^*, together with an internal function object F satisfying (1). Such a category is called a *closed* symmetric monoidal topological category.

2 Categories of $\mathcal{D}$-spectra

In any symmetric monoidal category, a monoid is an object R together with an associative product $\phi : R\square R \to R$ and a unit $\lambda : u \to R$. A monoid is commutative if ϕ is commutative up to coherent natural isomorphism.

Definition 2.1. *A* $\mathcal{D}$-spectrum *over R is a $\mathcal{D}$-space $X : \mathcal{D} \to \mathcal{T}$ together with natural continuous maps*

$$\sigma : X(d) \wedge R(e) \to X(d \Box e)$$

such that the composite

$$X(d) \cong X(d) \wedge S^0 \stackrel{1\wedge\lambda}{\to} X(d) \wedge R(u) \stackrel{\sigma}{\to} X(d \Box u) \cong X(d)$$

is the identity and the diagram

$$\begin{array}{ccc} X(d) \wedge R(e) \wedge R(f) & \xrightarrow{\sigma\wedge 1} & X(d\Box e) \wedge R(f) \\ \downarrow{\scriptstyle 1\wedge\phi} & & \downarrow{\scriptstyle \Box} \\ X(d) \wedge R(e \Box f) & \xrightarrow{\sigma} & X(d \Box e \Box f) \end{array}$$

commutes. Let $\mathcal{DS}_R$ denote the category of $\mathcal{D}$-spectra over R.

Remark 2.2. *Note that σ is the data of a map $X \overline{\wedge} R \to X \circ \Box$ of $\mathcal{D} \times \mathcal{D}$-spaces. In any symmetric monoidal category, a (right) R-module is than an object M together with $M \Box' R \to R$ which is associative and unital. Here $\Box'$ denotes the operation. By* (3)*, σ gives a map $X \wedge R \to X$. Unwinding definitions, we see that a $\mathcal{D}$-spectrum over R is the same as an R-module. See Proposition 1.10 of [MMSS].*

Example 2.3. *Consider the $\mathbb{N}$-spaces of Example 1.2. Define $S : \mathbb{N} \to \mathcal{T}$ by $S(n) = S^n$. Then S is a monoid. To see this, note that for X, Y in $\mathbb{N}\mathcal{T}$, the smash product $X \wedge Y$ is given by*

$$(X \wedge Y)(n) = \vee_{p=0}^{n} X_p \wedge Y_{n-p}.$$

We have $S \wedge S \to S$ given by the standard maps $S^p \wedge S^{n-p} \to S^n$.

$\mathbb{N}\mathcal{S}_S$ is the category of spectra constructed in Lecture 2. On the other hand, S is NOT a commutative monoid, because the map $S^d \wedge S^e \to S^e \wedge S^d$ flipping the factors is not the identity on S^n. This is the source of the difficulty in defining the smash product of spectra.

Theorem 2.4. *Let R be a commutative monoid in $\mathcal{DT}$. Then the category $\mathcal{DS}_R$ has a smash product $\wedge_R$ and an internal Hom F_R making $\mathcal{DS}_R$ a closed symmetric monoidal topological category.*

Proof. By Remark 2.2, we need to make the category of (right) R-modules into a closed symmetric monoidal category. The definitions can be borrowed from algebra. Given a right R-module X, and a left R-module Y, define $X \wedge_R Y$ to be the coequalizer in $\mathcal{D}$-spaces

$$X \wedge R \wedge Y \rightrightarrows X \wedge Y \to X \wedge_R Y.$$

Since R is commutative, we can identify right and left modules. Furthermore, $X \wedge_R Y$ inherits the structure of an R-module. R is the unit for $\wedge_R$.

Define $F_R(X, Y)$ to be the equalizer in $\mathcal{D}$-spaces

$$F_R(X, Y) \to F(X, Y) \rightrightarrows F(X \wedge R, Y).$$

Here, the two maps $F(X, Y) \to F(X \wedge R, Y)$ are as follows. One of them is pull-back by the map $X \wedge R \to R$ coming from the R-module structure on X. The other is adjoint to the composite

$$F(X, Y) \wedge X \wedge R \to Y \wedge R \to Y,$$

where the first map comes from the evaluation $F(X, Y) \wedge X \to Y$ ad the second map comes from the module structure on Y. □

Example 2.5. *(Symmetric spectra) Let Σ be as in Example 1.3. Let S be the Σ-space defined $S(n) = S^n$. S is a commutative monoid. To see this, note that for X, Y Σ-spaces, we have that $(X \wedge Y)(n) = \vee_{p=0}^{n} (\Sigma_n)_+ \wedge_{\Sigma_p \times \Sigma_{n-p}} X(p) \wedge Y(n-p)$. This can also be written $(X \wedge Y)(n) = \vee_{[p] \subset [n]} X(p) \wedge Y(n-p)$ where the wedge sum ranges over all subsets of $[n] = \{0, 1, \dots, n\}$. This gives a* twist *isomorphism $X \wedge Y \to Y \wedge X$ obtained by mapping the summand corresponding to $[p]$ to the summand corresponding to $[n] - [p]$ by the twist isomorphism $X(p) \wedge Y(n-p) \to Y(n-p) \wedge X(p)$ in spaces. (Note that this also allows us to see that the smash product on $\Sigma\mathcal{T}$ is commutative.) Given such a subset, the associated map $S^p \wedge S^{n-p} \to S^n$ is the corresponding permutation. This makes S commutative. This idea is due to Jeff Smith.*

$\Sigma\mathcal{S}_S$ is the category of symmetric spectra. Note the functor $\iota : \mathbb{N} \to \Sigma$ which sends n to n. This gives as un underlying spectrum in the sense of lecture 2 associated to any symmetric spectrum. The functor $\mathbb{U} : \Sigma\mathcal{S}_S \to \mathbb{N}\mathcal{S}_S$ sending a symmetric spectrum to its underlying spectrum admits a left adjoint prolongation functor *$\mathbb{P} : \mathbb{N}\mathcal{S}_S \to \Sigma\mathcal{S}_S$. For a based CW-complex A, we have $(\mathbb{P}(\Sigma^\infty A))(n) = A \wedge S^n$. Since $\mathbb{P}$ preserves colimits, this determines $\mathbb{P}$.*

Once we have a way to discuss the homotopy theory of $\mathbb{N}\mathcal{S}_S$ and $\Sigma\mathcal{S}_S$, we will want to see that the associated homotopy categories are the same. This is shown in [MMSS, Corollary 10.5]. We then have the desired smash product and function object on the stable homotopy category of Lecture 3.

Exercise 2.6. *For Y and X in based CW-complexes, show that $\mathbb{P}(\Sigma^\infty X) \wedge_S \mathbb{P}(\Sigma^\infty Y) \cong \mathbb{P}(\Sigma^\infty (X \wedge Y))$. (Hint: define a reasonable $X \wedge S$ (i.e. do the smash for each n), show this is $\mathbb{P}(\Sigma^\infty X)$, and then use the fact that $S \wedge_S S \cong S$.)*

Example 2.7. *(Orthogonal spectra) Let $\mathcal{I}$ be as in Example 1.4. The smash product on $\mathcal{I}$-spaces is given by*

$$(X \wedge Y)(V) = \vee_{W \subset V} X(W) \wedge Y(V - W),$$

where $V - W$ denotes the orthogonal complement of W in V. If you choose a particular p dimensional subspace V_p of V for every p, this can also be written

$$(X \wedge Y)(V) = \vee_p O(n)_+ \wedge_{O(p)\times O(n-p)} X(V_p) \wedge Y(V - V_p),$$

where n is the dimension of p. Let S be the $\mathcal{I}$ space which takes an inner product space V to S^V, where S^V denotes the one-point compactification of V. Then S is a commutative monoid in $\mathcal{IT}$. $\mathcal{IS}_S$ is the category of orthogonal spectra. Note the functor $\Sigma \to \mathcal{I}$ which takes n to $\mathbb{R}^n$ with the standard inner product allows us to associate an underlying symmetric spectrum to any orthogonal spectrum. There is an associated prolongation functor as well. The underlying spectrum functor and the prolongation functor again produce an equivalence of homotopy categories [MMSS, Corollary 10.5].

References

[MMSS] M.A. Mandell, J.P. May, S. Schwede, and B. Shipley *Model Categories of Diagram Spectra*, 1999.