Lecture 17: Application of Adams operations to finish the vector fields on spheres problem

3/6/15

1 Some computations of KO with Adams operations

Lemma 1.1. *ψ^k acts on $K^0(S^{2n})$ by k^n.*

Proof. Recall our calculation $K^0(\mathbb{CP}^n) \cong \mathbb{Z}[\mu]/(\mu^{n+1})$ where μ represents the class $\mathbb{L}_{\mathbb{C}} - 1$. From the cofiber sequence

$$\mathbb{CP}^{n-1} \to \mathbb{CP}^n \to S^{2n},$$

we have that the map $\mathbb{Z} \cong K^0(S^{2n}) \to K^0(\mathbb{CP}^n)$ has image μ^n. Since we saw in last lecture that $\psi^k(\mu) = (\mu+1)^k - 1$, we have $\psi^k(\mu^n) = ((\mu+1)^k - 1)^n = (k\mu + \ldots \mu^k)^n = k^n\mu^n$. This shows the lemma. $\square$

Tensoring a real vector bundle with $\mathbb{C}$ produces a complex vector bundle. This defines an operation $c : KO^0(X) \to K^0(X)$.

Forgetting the complex structure on a complex vector bundle produced a real vector bundle. This defines an operation $r : K^0(X) \to KO^0(X)$.

The map $rc : KO^0(X) \to KO^0(X)$ is multiplication by 2.

Lemma 1.2. *ψ^k acts on $\tilde{KO}^0(S^{4j}) \cong \mathbb{Z}$ by multiplication by k^{2j}.*

Proof. The map $c : \mathbb{Z} \cong KO^0(S^{4j}) \to K^0(S^{4j}) \cong \mathbb{Z}$ is is non-zero because rc is multiplication by 2. Therefore it is multiplication by some non zero integer n. Since c commutes with the Adams operations (see last lecture), this lemma follows from the previous. $\square$

Lemma 1.3. *$\tilde{KO}(\mathbb{RP}^n)$ is cyclic, generated by $\mathbb{L}-1$, and has order a_n, where $a_n = 2^{x_n}$ with x_n equal to the number of q such that there is a non-zero entry in the $E^2_{q,-q} = \tilde{\mathrm{H}}^q(\mathbb{RP}^n, \pi_{-q}KO)$ entry of the Atiyah-Hirzebruch spectral sequence.*

Furthermore, ψ^{2k+1} acts as the identity for all integers k.

Proof. The group structure of $\tilde{KO}(\mathbb{RP}^n)$ is due to Bott and Shapiro. Here is a summary of the argument given in [A2]. First use the projection $\pi : \mathbb{RP}^{2n+1} \to \mathbb{CP}^n$ to compare $\tilde{K}^0(\mathbb{CP}^n)$ and $\tilde{K}^0(\mathbb{RP}^{2n+1})$. The map π induces a map of AHSS's, and both of these collapse. The fact that the powers of u in $K^0(\mathbb{CP}^n)$ are non-trivial can be used to show that $\tilde{K}(\mathbb{RP}^n)$ has the maximal possible order and is generated by the pull-back of u. Once equipped with the cyclic structure of $\tilde{K}(\mathbb{RP}^n)$, one compares $\tilde{K}(\mathbb{RP}^n)$ and $\tilde{KO}(\mathbb{RP}^n)$ using c. For example, if n is congruent to 6, 7, or 8 mod 8, then

$$c : \tilde{KO}(\mathbb{RP}^n) \to \tilde{K}(\mathbb{RP}^n)$$

is an isomorphism [A2, Lemma 7.5].

To show that ψ^{2k+1} acts as the identity, it is therefore sufficient to see that $\mathbb{L}^{2k+1} \cong \mathbb{L}$. Thus it is sufficient to see that $\mathbb{L}^2 \cong \underline{\mathbb{R}}$, where $\underline{\mathbb{R}}$ denotes the trivial bundle. A real line bundle is formed by taking open sets U_i of your space, forming $U_i \times \mathbb{R}$, and gluing along maps $\phi_{ij} : U_i \cap U_j \to GL_1\mathbb{R}$. You can replace $GL_1\mathbb{R} \cong \mathbb{R}^*$ by its compact form $O_1\mathbb{R} = \{-1, 1\}$. When you take the kth tensor power of a line bundle you replace ϕ_{ij} by $x \mapsto (\phi_{ij}(x))^k$. Since -1 and 1 both square to 0, we have that $\mathbb{L} \otimes \mathbb{L} \cong \underline{\mathbb{R}}$, as claimed. □

Lemma 1.4. *Suppose $0 < 4j \leq n$. Then $\tilde{KO}(\mathbb{RP}^n_{4j}) \cong \mathbb{Z} \oplus \mathbb{Z}/2^x$, where $x = x_n - x_{4j}$.*

Furthermore, the map $S^{4j} \to \mathbb{RP}^n_{4j}$ induces a surjection on $\tilde{KO}$,

$$\tilde{KO}(\mathbb{RP}^n_{4j}) \to \tilde{KO}(S^{4j}) \cong \mathbb{Z}.$$

Furthermore, $\mathbb{Z}/2^x$ is in the kernel of $\tilde{KO}(\mathbb{RP}^n_{4j}) \to \tilde{KO}(\mathbb{RP}^n)$.

A proof is in [A2, Theorem 7.4].

2 Last step of the vector field problem

Let m be a positive integer and express m as $m = (2a+1)2^b$, $b = c + 4d$ with $0 \leq c \leq 3$. Let $\rho(m) = 2^c + 8d$.

For m a positive positive integer, and $n \geq m$, we have $\mathbb{RP}^n_m = \mathbb{RP}^n/\mathbb{RP}^{m-1}$.

Note that we have a map $i : S^m \cong \mathbb{RP}^m_m \to \mathbb{RP}^n_m$.

Theorem 2.1. *[A2, Theorem 1.2] Let m be a positive integer. Then there does not exist a map $f : \mathbb{RP}^{m+\rho(m)}_m \to S^m$ such that $fi : S^m \to S^m$ has degree 1.*

Adams proves this theorem in the case where m is divisible by 8 in [A2]. When m is not divisible by 8, we have $d = 0$, and there is an argument with Steenrod squares (Steenrod squares are cohomology operation on $H\mathbb{Z}/2$) which proves the theorem [JW].

Proof. Assume m divisible by 8. Then $\tilde{KO}(\mathbb{RP}^{m+\rho(m)}_m) \cong \mathbb{Z} \oplus \mathbb{Z}/(2^{b+1})$ by Lemma 1.4 and a little case by case arithmetic. Here as above, b is defined by $m = (2a+1)2^b$. The cofiber sequence

$$\mathbb{RP}^{m-1} \to \mathbb{RP}^{m+\rho} \to \mathbb{RP}^{m+\rho(m)}_m$$

induces a sequence

$$\tilde{KO}\mathbb{RP}^{m+\rho(m)}_m \to \tilde{KO}\mathbb{RP}^{m+\rho} \to \tilde{KO}\mathbb{RP}^{m-1}$$

which is exact in the middle.

By Lemma 1.3, we have that the kernel of

$$\tilde{KO}\mathbb{RP}^{m+\rho} \to \tilde{KO}\mathbb{RP}^{m-1}$$

has order at least 2^{b+2}. (The number of non-zero entries along the diagonal of AHSS between $m+1$ and $m+\rho$, including the ends, was already given to be $b+1$, and then there is one more.) Thus the order of the image of

$$\tilde{KO}\mathbb{RP}^{m+\rho(m)}_m \to \tilde{KO}\mathbb{RP}^{m+\rho}$$

must be at least 2^{b+2}.

Suppose for the sake of contradiction, that we had a map f as in the statement of the theorem. Then we have a map

$$\tilde{KO}(S^m) \xrightarrow{f^*} \tilde{KO}(\mathbb{RP}^{m+\rho(m)}_m).$$

By Lemma 1.4, the image of the composition

$$\tilde{KO}(S^m) \xrightarrow{f^*} \tilde{KO}\mathbb{RP}^{m+\rho(m)}_m \to \tilde{KO}\mathbb{RP}^{m+\rho}$$

equals the image of

$$\tilde{KO}\mathbb{RP}^{m+\rho(m)}_m \to \tilde{KO}\mathbb{RP}^{m+\rho}.$$

So the image of

$$\tilde{KO}(S^m) \to \tilde{KO}\mathbb{RP}^{m+\rho}$$

must have order at least 2^{b+2}.

Let α be a generator of $\tilde{KO}(S^m)$, and let $\overline{\alpha}$ denote the image of α in $\tilde{KO}\mathbb{RP}^{m+\rho}$. Since ψ^3 is the identity on $\tilde{KO}\mathbb{RP}^{m+\rho}$, we have that $\psi^3\overline{\alpha} - \overline{\alpha} = 0$. Thus the image of $\psi^3(\alpha) - \alpha$ is zero 0. By 1.2, we have that $\psi^3(\alpha) - \alpha = (3^{(2a+1)2^{b-1}} - 1)\alpha$. Combining with the previous gives that the image of $(3^{(2a+1)2^{b-1}} - 1)\alpha$ in $\tilde{KO}\mathbb{RP}^{m+\rho}$ is 0. Note that $3^{(2a+1)2^{b-1}} - 1 = (8+1)^{(2a+1)2^{b-2}} - 1$, which is of the form $2^{b+1}k$ for k an odd number. Since $\tilde{KO}\mathbb{RP}^{m+\rho}$ is a group whose order is a power of 2, it follows that the image of $2^{b+1}\alpha$ in $\tilde{KO}\mathbb{RP}^{m+\rho}$ is 0. This contradicts the fact that the image must have order at least 2^{b+2}

□

References

[A2] J.F. Adams, *Vector Fields on Spheres*, Annals of Mathematics, second series **75** No. 3, 1962, pp. 603-632.

[H] Michael Hopkins, *Stable homotopy theory (course notes)*.

[JW] I. James and J.H.C. Whitehead *Vector fields on the n-sphere*,Proc. Nat. Acad. Sci. USA, 37 (1951), 58-63.