Lecture 16: Adams operations and K-theory of projective spaces

3/4/15

1 Adams Operations

Let X be a finite CW-complex. Adams operations are certain homomorphisms $\psi^k : K^0(X) \to K^0(X)$, or $KO^0(X) \to KO^0(X)$. Moreover, they are ring homomorphisms. The tensor product $\otimes$ of vector bundles determines an operation $\otimes$ on K-theory, because the defining equivalence relation is respected, making $K^0(X)$ and $KO^0(X)$ into rings. In fact, K and KO are ring spectra. We'll get back to this, but for now, let's say that a spectrum R is a ring spectrum if R is equipped with a map $R \wedge R \to R$, satisfying certain properties, and then $R^*(X)$ inherits a cup-product, making $R^*(X)$ into a graded ring and $R^0(X)$ into a ring. The multiplication is said to be commutative if $R \wedge R \stackrel{\text{flip}}{\to} R \wedge R \to R$ is the same map as $R \wedge R \to R$, in other words, we have a map $(R \wedge R)/\mathbb{Z}/2 \to R$. We can loosen this condition up by only asking that there is a map $((E\mathbb{Z}_2)_+ \wedge (R^{\wedge 2}))/\mathbb{Z}/2 \to R$, where $E\mathbb{Z}/2$ is the universal covering space of $\mathbb{RP}^\infty$. Let Σ_n denote the symmetric group on n-letters, let $E\Sigma_n$ denote a contractible space with a free Σ_n-action, and let $D_n = ((E\Sigma_n)_+ \wedge (R^{\wedge n}))/\Sigma_n$. The data of maps $D_n R \to R$ makes R into an example of a *structured ring spectrum*, and under some restrictions gives *power operations* in R-cohomology. Adam's operations on K-theory can be constructed along these lines. They were first constructed by Adams [A2] to solve the Vector Field problem, and they have the following concrete definition in terms of vector bundles.

For the following theorem, let K denote either K or KO. In the case of K, the vector bundles are complex, and in the case of KO, the vector bundles are real. When we need to distinguish between the two, we'll put a subscript $\mathbb{R}$ or $\mathbb{C}$.

Theorem 1.1. *There are natural homomorphisms $\psi^k : K(X) \to K(X)$ for $k \in \mathbb{Z}$ satisfying the following properties:*

1. *ψ^k is a ring homomorphism.*
2. *For any line bundle L, $\psi^k L = L^{\otimes k}$.*
3. *$\psi^1 = id$. ψ^0 assigns to every bundle the trivial bundle with the same rank. $\psi_{\mathbb{C}}^{-1}$ is complex conjugation (explained in proof) and $\psi_{\mathbb{R}}^{-1}$ is the identity.*
4. *$\psi^k \psi^l = \psi^{kl}$*
5. *$c\psi_{\mathbb{R}}^k = \psi_{\mathbb{C}}^k c$ where c denotes complexification.*

An element of K-theory is a difference of vector bundles, so ψ^k is determined by its value on vector bundles. Since (2) dictates the behavior of ψ^k on line bundles, we would like to think of an vector bundle as a sum of line bundles. The tool for doing this is:

Theorem 1.2. *(Splitting principle) Given a vector bundle $E \to X$, there is a finite CW complex $P(E)$ and a map $p : P(E) \to X$ such that the induced map $p^* : K^*(X) \to K^*(P(E))$ is injective and p^*E splits as a sum of line bundles.*

We're going to skip the proof, but see [H2, 2.3 p 66].

The *elementary symmetric polynomials* $\sigma_j(t_1, \ldots, t_r)$ are defined by $\prod_{j=1}^r (x + t_j) = \sum_{i=0}^r \sigma_{r-i}(t_1, \ldots, t_r)x^i$. It is an algebraic fact that the elementary symmetric polynomials generate $\mathbb{Z}[t_1, \ldots, t_r]^{\Sigma_r}$. Thus, there exists a polynomial s_k such that

$$s_k(\sigma_1, \ldots, \sigma_r) = t_1^k + \ldots + t_r^k.$$

A recursive formula for s_k is

$$s_k = \sigma_1 s_{k-1} e \sigma_2 s_{k-2} + \ldots + (-1)^{k-1} k \sigma k.$$

Note this implies that s_k only depends on $\sigma_1, \ldots, \sigma_k$.

Proof. (of Theorem 1.1. cf. [A2, Theorem 5.1] and [H2, Theorem 2.20]) First take $k > 0$. Let E be a vector bundle. Let $\wedge^n E$ denote the n-th exterior power of E with itself. Define

$$\psi^k(E) = s_k(E, \wedge^2 E, \ldots, \wedge^k E).$$

Note that this definition commutes with pull-backs along maps $X \to Y$ of finite CW complexes, and agrees with (2). Furthermore, if $E = L_1 \oplus \ldots \oplus L_r$, then $\psi^k(E) = L_1^k \oplus \ldots \oplus L_r^k$ by construction, because the exterior powers can be identified with the symmetric polynomials. By the splitting principle, it follows that if E_1 and E_2 are two vector bundles then $\psi^k(E_1 \oplus E_2) = \psi^k(E_1) \oplus \psi^k(E_2)$. It follows that ψ^k determines a natural additive group homomorphism on $K^0(X)$.

To show that ψ^k is a ring homomorphism, it suffices to check that $\psi^k(E_1 \otimes E_2) = \psi^k(E_1)\psi^k(E_2)$, where the E_i are sums of line bundles. In this case $E_1 \otimes E_2 = \sum_{i,j} L_i \otimes L'_j$ and $\psi^k(E_1 \otimes E_2) = \sum_{i,j} L_i^{\otimes k} \otimes (L'_j)^{\otimes k} = (\sum_i L_i^{\otimes k}) \otimes (\sum_j (L'_j)^{\otimes k}) = \psi^k(E_1) \otimes \psi^k(E_2)$, showing 1.

The first claim of (3), follows from the fact that $s_1(\sigma_1) = \sigma_1$. We use the remaining claims of (3) to define ψ^k. Here complex conjugation of a complex vector bundle means that the transition functions from open sets to $\mathrm{GL}_n \mathbb{C}$ are composed with complex conjugation $\mathrm{GL}_n \mathbb{C} \to \mathrm{GL}_n \mathbb{C}$.

(4) holds for line bundles by construction. By the splitting principle, the case of line bundles implies the result on K-theory.

It again suffices to show (5) for line bundles. It thus suffices to show the tensor product commutes with complexification. This is true because complexification uses the same transition functions from open sets to $\mathbb{R}^*$, just considering elements of $\mathbb{R}^*$ to be elements of $\mathbb{C}^*$, and tensor product multiplies the transition functions of the two line bundles. □

Now let K-theory denote complex K theory as usual. Any complex line bundle L over X is the pull-back of $\mathbb{L}_{\mathbb{C}} \to \mathbb{CP}^\infty$ by some map $\phi : X \to \mathbb{CP}^\infty$. Moreover, $\mathrm{H}^*(\mathbb{CP}^\infty; \mathbb{Z}) = \mathbb{Z}[[z]]$, where z is in H^2 and is compatible with the Poincaré dual of the point in $\mathbb{CP}^1 \cong S^2$. The first chern class $\mathrm{ch}^1 L \in \mathrm{H}^2(X, \mathbb{Z})$ of a line bundle L over X can be defined by $\mathrm{ch}^1 L = \phi^*(-z)$.

Proposition 1.3. *There is a ring homomorphism* $\mathrm{ch} : K^0(X) \to \mathrm{H}^*(X, \mathbb{Q})$ *such that*

1. $\mathrm{ch}\, L = e^z$ *for a line bundle* L *and where* z *denotes the first Chern class of* L*. Here the expression* e^z *means the element in the graded ring* $\mathrm{H}^*(X, \mathbb{Q})$ *given by* $e^z = \sum_j z^j/j!$.
2. $\mathrm{ch}^q : \tilde{K}^0(S^{2q}) \to \mathrm{H}^{2q}(S^{2q}; \mathbb{Q})$ *maps* $\tilde{K}(S^{2q})$ *isomorphically onto the image of* $\mathrm{H}^{2q}(S^{2q}, \mathbb{Z})$.

For the existence of ch, we set $\mathrm{ch}(L_1 \oplus L_2 \oplus \ldots \oplus L_r) = \sum_j e^{\mathrm{ch}^1(L_i)}$, and more generally rely on the splitting principle. For (2), there is interesting work to be done, but we're going to skip it. See [AT, Proposition 2.2]. Atiyah credits this sort of result to Bott.

2 Calculations

Let $\mu = \mathbb{L}_{\mathbb{C}} - 1$ in $K^0(\mathbb{CP}^n)$.

Theorem 2.1. *(Adams [A2, Theorem 7.2])* $K^0(\mathbb{CP}^n) \cong \mathbb{Z}[\mu]/(\mu^{n+1})$, *and* $\psi^k(\mu) = (\mu+1)^{\otimes k} - 1$.

For k negative, $(1+\mu)^k$ denotes the binomial expansion $(1+\mu)^k = \sum_{j=0}^{\infty} \binom{k}{j}\mu^j = \sum_{j=0}^{n} \binom{k}{j}\mu^j$.

Proof. We first show $K^0(\mathbb{CP}^n) \cong \mathbb{Z}[\mu]/(\mu^{n+1})$. By the Atiyah-Hirzebruch spectra sequence, there is an additive isomorphism between $K^0(\mathbb{CP}^n)$ and $\mathbb{Z}^{n+1}$. We need to show that $1, \mu, \ldots, \mu^n$ is a basis. The result holds for $n = 0$. Inductively suppose the result for $n-1$. Then $\mu^n \in K^0(\mathbb{CP}^n)$ maps to 0 under

$$K^0(\mathbb{CP}^n) \to K^0(\mathbb{CP}^{n-1}).$$

Thus μ^n comes from an element μ^n of $K^0(\mathbb{CP}^n/\mathbb{CP}^{n-1})$ by the LES of the pair $(\mathbb{CP}^n, \mathbb{CP}^{n-1})$. Note that $\operatorname{ch}\mu = \operatorname{ch}\mathbb{L}_{\mathbb{C}} - 1 = \sum_{j=1} y^n/n!$, where y generates $\mathrm{H}^2(\mathbb{CP}^n, \mathbb{Z})$. Thus $\operatorname{ch}\mu^n = y^n$. Since ch is natural, we also have "the same" equality in $\mathbb{CP}^n/\mathbb{CP}^{n-1} \cong S^{2n}$. So by Proposition 1.3 (2), we have that μ^n comes from a generator for $K^0(S^{2n})$. Thus μ^n represents a generator of $E_2^{-n,n}$ of the Atiyah-Hirzebruch spectral sequence. It follows that $1, \mu, \ldots, \mu^n$ is a basis as claimed.

By Proposition 1.1 (2), $\psi^k(\mu+1) = (\mu+1)^{\otimes k}$. By Proposition 1.1 (1), we have

$$\psi^k(\mu) = \psi^k(\mu+1) - \psi^k 1 = (\mu+1)^{\otimes k} - 1.$$

□

References

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[H2] Allen Hatcher, *Vector Bundles and K-theory*, book on-line.