Lecture 15: Vector fields on spheres

2/25/15

1 (Real) Stiefel Manifolds and projective space

Let $V_{n,k}$ be the space of ordered k-tuples of orthonormal vectors in $\mathbb{R}^n$. Note that $V_{n,k} \cong SO(n)/SO(n-k)$. $V_{n,k}$ is called a *Stiefel manifold.* Singling out the first vector of the k-tuple gives a map $e_1 : V_{n,k} \to S^{n-1}$.

Remark 1.1. *S^{n-1} has $k-1$ vector fields if and only if $V_{n,k} \to S^{n-1}$ admits a section.*

So we wish to understand the topology of $V_{n,k}$. Note that we have a fiber bundle $SO(n-1) \to SO(n) \to S^{n-1}$. If we had $SO(n) \cong SO(n-1) \times S^{n-1}$, then by induction $SO(n) \cong S^{n-1} \times S^{n-2} \times \ldots \times S^1$. Now, it is not true that $SO(n) \cong SO(n-1) \times S^{n-1}$, but the homology $\mathrm{H}_*(V_{n,k}, \mathbb{Z}/2)$ of the Stiefel manifold is the exterior algebra over $\mathbb{Z}/2$ on generators x_i of degree i, denoted $E[x_{n-1}, \ldots, x_1]$. (This is just a memory aid.) To see some of the structure of $V_{n,k}$, consider the map

$$\mathbb{RP}^{n-1} \to SO(n)$$

defined as follows. Fix a line ℓ_0 in $\mathbb{R}^n$. Consider $\mathbb{RP}^{n-1}$ to be the set of lines through the origin in $\mathbb{R}^n$. For $\ell \in \mathbb{RP}^n$, let $R_\ell : \mathbb{R}^n \to \mathbb{R}^n$ be reflection through the plane perpendicular to ℓ. By sending ℓ to $R_\ell R_{\ell_0}$, we have the desired map $\mathbb{RP}^{n-1} \to SO(n)$. Note that $R_\ell R_{\ell_0}$ fixes the hyperplane perpendicular to the plane spanned by ℓ and ℓ_0, and in the plane spanned by ℓ and ℓ_0 is a rotation. If ℓ_0 is the last standard basis element e_n of $\mathbb{R}^n$ and $SO(n) \to S^{n-1}$ is the evaluation on the first standard basis element e_1, we have that all lines perpendicular to e_1 are sent to $e_1 \in S^{n-1}$. We can arrange for the diagram

$$\begin{array}{ccc} & & SO(n) \\ & \nearrow & \downarrow \\ \mathbb{RP}^{n-1} & \longrightarrow & \mathbb{RP}^{n-1}/\mathbb{RP}^{n-2} \cong S^{n-1} \end{array}$$

to be commutative.

We then have a commutative diagram of spaces

$$\begin{array}{ccc} SO(n-k) & \longrightarrow & SO(n) \\ \uparrow & & \uparrow \\ \mathbb{RP}^{n-k-1} & \longrightarrow & \mathbb{RP}^{n-1} \end{array} .$$

Since the composition $SO(n-k) \to SO(n) \to V_{n,k}$ is the constant map to the k-tuple of the first standard basis elements of $\mathbb{R}^n$, we have that the composition $\mathbb{RP}^{n-1} \to SO(n) \to V_{n,k}$ is constant on $\mathbb{RP}^{n-k-1}$. We thus have a map

$$\iota : \mathbb{RP}^{n-1}_{n-k} = \mathbb{RP}^{n-1}/\mathbb{RP}^{n-k-1} \to V_{n,k}$$

Proposition 1.2. *There is a cell-structure on $V_{n,k}$ such that $\mathbb{RP}^{n-1}_{n-k}$ is a subcomplex. Furthermore, $V_{n,k}/\mathbb{RP}^{n-1}_{n-k}$ is $2(n-k)$-connected.*

This is shown in [J, §3 p. 24-25]. ($d = 1$ in the case of real projective space and Stiefel manifolds; James is also doing complex and quaternionic Stiefel manifolds.)

Corollary 1.3. *(James) Suppose that $n-1 \leq 2(n-k)$. There is a section of $V_{n,k} \to S^{n-1}$ if an only if there is a map $S^{n-1} \to \mathbb{RP}^{n-1}_{n-k}$ such that the composition $S^{n-1} \to \mathbb{RP}^{n-1}_{n-k} \to \mathbb{RP}^{n-1}_{n-k}/\mathbb{RP}^{n-2}_{n-k} \cong S^{n-1}$ is homotopic to the identity.*

The condition that there is a map $S^{n-1} \to \mathbb{RP}^{n-1}_{n-k}$ is called *reducible* by James and Adams. It is equivalent to

$$\mathbb{RP}^{n-1}_{n-k} \cong \mathbb{RP}^{n-2}_{n-k} \vee S^{n-1} \tag{1}$$

where the homotopy equivalence takes place in the category of spaces and is the identity on the sub complex $\mathbb{RP}^{n-2}_{n-k}$. In other words, the top cell of $\mathbb{RP}^{n-1}_{n-k}$ has trivial attaching map. Equivalently, the top cell of $\mathbb{RP}^{n-1}$ has an attaching map factoring through the $(n-k-1)$-skeleton.

Proof. Suppose there is a map $s : S^{n-1} \to \mathbb{RP}^{n-1}_{n-k}$ as in the statement of the corollary. Choose a homotopy $H : S^{n-1} \times I \to S^{n-1}$ between the identity on S^{n-1} and $e_1 \iota s$. We obtain a commutative diagram

$$\begin{array}{ccc} S^{n-1} & \xrightarrow{\iota s} & V_{n,k} \\ {\scriptstyle x \mapsto x \times 0} \downarrow & & \downarrow {\scriptstyle e^1} \\ S^{n-1} \times I & \xrightarrow{H} & S^{n-1} \end{array} .$$

It is a fact that e_1 is a fiber-bundle, whence a fibration, and that the property of being a fibration implies that there is $\tilde{H}: S^{n-1} \times I \to V_{n,k}$ which when added to the diagram, keeps the diagram commutative. The restriction of $\tilde{H}$ to $S^{n-1} \times 1$ gives the desired section.

Suppose there is a section of $V_{n,k} \to S^{n-1}$. Since $n-1 \leq 2(n-k)$, the $n-1$ skeleton of $V_{n,k}$ can be taken to be $\mathbb{RP}^{n-1}_{n-k}$. It follows that any map $S^{n-1} \to V_{n,k}$, and, in particular, the section $s : S^{n-1} \to V_{n,k}$, is homotopic to a map $S^{n-1} \to \mathbb{RP}^{n-1}_{n-k}$. □

2 Dualizing

Recall that the Spanier-Whitehead dual takes cofiber sequences to cofiber sequences. Also, note that the dual of a split cofiber sequence therefore is also a split cofiber sequence because the the Spanier-Whitehead dual is a functor. So *in the stable homotopy category, the condition* (1) *that the top cell of* $\mathbb{RP}^{n-1}_{n-k}$ *splits off is equivalent to the statement that the dual of* $\mathbb{RP}^{n-1}_{n-k}$ *has its bottom cell split off.*

Atiyah duality computes this dual. Namely, recall that $\mathbb{RP}^a_b = \mathrm{Th}(\mathbb{RP}^{a-b}, b\mathbb{L})$, where $\mathbb{L}$ denotes the tautological bundle. By Atiyah duality, $\mathbb{D}\,\mathrm{Th}(X,V) \cong \mathrm{Th}(X, -T_X - V)$. Thus $\mathbb{D}\mathbb{RP}^a_b \cong \mathrm{Th}(X, -T_X - b\mathbb{L}) \cong \mathrm{Th}(X, -(a-b+1)\mathbb{L} + \underline{\mathbb{R}} - b\mathbb{L}) \cong \Sigma\,\mathrm{Th}(X, -(a+1)\mathbb{L}) \cong \Sigma\mathbb{RP}^{-(b+1)}_{-(a+1)}$. (Over $\mathbb{R}$, there is an equivalence between $\mathbb{L}$ and $\mathbb{L}^*$.) In summary,

$$\mathbb{D}\mathbb{RP}^a_b \cong \Sigma\mathbb{RP}^{-(b+1)}_{-(a+1)}.$$

In the Atiyah-duality lecture, we computed that $\mathbb{D}\mathbb{RP}^n_+ \cong \Sigma\mathbb{RP}^{-1}_{-(n+1)}$. It is compatible to therefore define $\mathbb{RP}^a_0 = \mathbb{RP}^a_+$. This spectrum clearly splits off the bottom cell. By James periodicity, we have that $\Sigma^{-n}\mathbb{RP}^{n+k}_n$ only depends on n modulo a_k, where a_k is as in S. Kolay's talk, i.e. a_k is the dimension of the smallest representation of C^+_k, or equivalently, the number of q such that there is a $\mathbb{Z}/2$ in $E^2_{q,-q} = \tilde{\mathrm{H}}(\mathbb{RP}^k, \pi_{-q}KO)$. (For example $a_8 = 16$.) Thus, we have that $\mathbb{RP}^{-ra_k+k}_{-ra_k}$ has its bottom cell split off. Thus $\mathbb{RP}^{ra_k-1}_{ra_k-1-k}$ has its top cell split off. *If this splitting were unstable* (we have that it exists stably), we could conclude that S^{ra_k-1} has k vector fields, at least provided r is large enough so we can apply Corollay 1.3. Note that the statement that S^{ra_k-1} has k vector fields is indeed true by what we know about Clifford algebras, so this is looking consistent. This discussion is meant to give evidence that the vector field problem is in stable homotopy theory. Let's now get on to Adams's proof.

3 Reducing to the stable range

Adams proves the "dual statement" to (1). Namely:

Theorem 3.1. *(Adams [A2, Theorem 1.2]) Let $\rho(n) = c + 4d$ where $n = (2a+1)2^b$, $b = c+4d$, and a,b,c,d are integers with $0 \le c \le 3$. There is no map $f : \mathbb{RP}^{n+\rho(n)}/\mathbb{RP}^{n-1} \to S^n$ such that the composite*

$$S^n \cong \mathbb{RP}^n/\mathbb{RP}^{n-1} \to \mathbb{RP}^{n+\rho(n)}/\mathbb{RP}^{n-1} \xrightarrow{f} S^n$$

has degree 1.

The conclusion is equivalent to the statement

$$\mathbb{RP}^{n+\rho(n)}/\mathbb{RP}^{n-1} \cong S^n \vee \mathbb{RP}^{n+\rho(n)}/\mathbb{RP}^n,$$

where this homotopy equivalence is in the category of spaces.

The proof of this uses Adams operations on K-theory, and we'll do it next time. Now, let's see that Theorem 3.1 proves:

Theorem 3.2. *(Adams) There do not exist $\rho(n)$ vector fields on S^{n-1}.*

The problem is that dualizing is an equivalence in spectra, and were are looking to show that that there does not exist (1) in spaces. When you know something is true in spectra and you wish it to be true in spaces, it is sometimes possible to reduce to the stable range, and this is what Adams does.

Lemma 3.3. *(James) If S^{n-1} admits $\rho(n)$ vector fields, then so does S^{pn-1} for any integer p.*

We omit the proof of this lemma. See [J2, Cor 1.4]

Proof. (of Theorem 3.2) Suppose the contrary. Then S^{pn-1} admits $\rho(n)$ vector fields. Provided p is sufficiently large, we have by Corollary 1.3 that there is a map $S^{pn-1} \to \mathbb{RP}^{n-1}_{n-(\rho(n)+1)}$ giving an unstable weak equivalence

$$S^{pn-1} \vee \mathbb{RP}^{pn-2}_{n-(\rho(n)+1)} \cong \mathbb{RP}^{pn-1}_{n-(\rho(n)+1)}.$$

Therefore in spectra, we have a weak equivalence

$$S^{-pn} \vee \mathbb{RP}^{-pn+\rho(n)}_{-pn+1} \cong \mathbb{RP}^{-pn+\rho(n)}_{-pn}$$

by taking Spanier-Whitehead duals. After suspension and applying James periodicity, we have

$$S^{qr-pn} \vee \mathbb{RP}^{qr-pn+\rho(n)}_{qr-pn+1} \cong \mathbb{RP}^{qr-pn+\rho(n)}_{qr-pn}$$

in spectra for any q, where $r = a_{\rho(n)}$. Set $m = qr - pn$.

Recall that a map of spectra from (the suspension spectrum of) an $m+\rho(n)$-dimensional CW complex to a complex which is at least about $2(m+\rho(n))$-connected is in fact a map of spaces because $[\Sigma^\infty Y, \Sigma^\infty X] = \varinjlim_l [S^l Y, \Sigma^l X]$ and all the transition maps are isomorphisms under these assumptions.

It follows that for m sufficiently large, we have a map of spaces $\mathbb{RP}^{m+\rho(n)}_m \to S^m$ inducing a homotopy equivalence of spaces

$$\mathbb{RP}^{m+\rho(n)}_m \cong \mathbb{RP}^{m+\rho(n)}_{m+1} \vee S^m.$$

We can choose q divisible by $2n$ and p odd, and ensure that $\rho(n) = \rho(m)$. This contradicts Theorem 3.1.

□

References

[A] J.F. Adams, *Stable Homotopy and Generalized Homology* Chicago Lectures in Mathematics, The University of Chicago Press, 1974.

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[J2] I. James *Whitehead products and vector-fields on spheres*,Proc. Cambridge Philos. Soc, 53 (1957) 817-820.