Lecture 14: K-theory, KO-theory, and James periodicity

2/20/15

1 Complex (topological) K-theory

Let X be a CW-complex, and suppose X is finite. Let $\mathrm{Vect}(X)$ denote the set of isomorphism classes of complex vector bundles on X. The Whitney sum gives an operation $\oplus$ on $\mathrm{Vect}(X)$.

Definition 1.1. *$K^0(X)$ is the initial group receiving a map from $\mathrm{Vect}(X)$ which sends $\oplus$ to the group operation on $K^0(X)$.*

More explicitly, $K^0(X)$ consists of pairs $(V, W) \in \mathrm{Vect}(X)^2$, subject to the equivalence $(V, W)\tilde{(}V', W')$ iff $V \oplus W' \oplus U \cong V' \oplus W \oplus U$ for some $U \in \mathrm{Vect}(X)$. The operation $\oplus$ extends to a group operation $(V, W) \oplus (V', W') = (V \oplus V', W \oplus W')$. We think of (V, W) as "$V \ominus W$."

Proposition 1.2.

1. *For $V, W \in \mathrm{Vect}(X)$, we have $V = W$ in $K^0(X)$ if and only if there exists n such that $V \oplus \underline{\mathbb{C}}^n \cong W \oplus \underline{\mathbb{C}}^n$, where $\underline{\mathbb{C}}$ denotes the trivial bundle on X.*
2. *Every element of $K^0(X)$ can be represented as $(V, \underline{\mathbb{C}}^n)$.*

Proof. If $V = W$ in $K^0(X)$, then there exists $U \in \mathrm{Vect}(X)$ such that $V \oplus U \cong W \oplus U$. It follows from compactness of X, that we may choose a finite cover so that over each open of the cover U is trivialized. A partition of unity argument then gives us an injection $U \to \underline{\mathbb{C}}^N$ for some large N. Choosing an inner product on $\underline{\mathbb{C}}^N$, we see that this injection is split. (In fact any injection of topological vector bundles is split). The kernel of this splitting is a vector bundle U' such that $U \oplus U' \cong \underline{\mathbb{C}}^N$. By taking the Whitney sum of both sides of $V \oplus U \cong W \oplus U$ with U', we see 1.

Now take (V, W) in $K^0(X)$. By the same reasoning, we may choose W' such that $W \oplus W' \cong \underline{\mathbb{C}}^N$. Then $(V, W) \cong (V \oplus W', \underline{\mathbb{C}}^N)$, showing the second claim.

□

Corollary 1.3. *If V,W in* $\mathrm{Vect}(X)$ *determine the same class in $K^0(X)$, then* $\mathrm{Th}(V) \cong \mathrm{Th}(W)$ *in the stable homotopy category.*

In the previous corollary, $\mathrm{Th}(V)$ denotes $\Sigma^\infty \mathrm{Th}(V)$.

Proof. Use 1 and the fact that $\mathrm{Th}(V \oplus \underline{\mathbb{C}}^n) \cong \Sigma^{2n} \mathrm{Th}(V)$. □

We may furthermore define the Thom spectrum $\mathrm{Th}(V, W)$ for any $(V, W) \in K^0(X)$ by choosing $(V', \underline{\mathbb{C}}^n)$ such that $(V, W) = (V', \underline{\mathbb{C}}^n)$, and setting $\mathrm{Th}(V, W) = \Sigma^{-2n} \mathrm{Th}(V')$. This is well-defined because if $(V_1, \underline{\mathbb{C}}^{n_1}) = (V_2, \underline{\mathbb{C}}^{n_2})$ in $K^0(X)$, we have $V_1 \oplus \underline{\mathbb{C}}^{n_2} \cong V_2 \oplus \underline{\mathbb{C}}^{n_1}$, whence $\Sigma^{2n_2} \mathrm{Th}(V_1) \cong \Sigma^{2n_1} \mathrm{Th}(V_2)$.

This proves some facts we were assuming before.

Given a map of spaces $X \to Y$, the pull-back of vector bundles gives a group homomorphism $K^0(Y) \to K^0(X)$. In fact K^0 extends to a generalized cohomology theory, represented by a spectrum K. You could start defining this generalized cohomology theory. For example,

$$K^{-n}(X) = \tilde{K}^0(S^n \wedge (X_+))$$

for all positive n. Perhaps it's better to write $\tilde{K}^{-n}X = \tilde{K}^0(S^n \wedge X)$. Here, $\tilde{K}^0(X)$ is the reduced K^0, given either as the subgroup of $K^0(X)$ consisting of (V, W) where V and W have the same rank, or as $\mathrm{Vect}/\sim$ where $V \sim W$ iff there are n and m such that $V \oplus \underline{\mathbb{C}}^n = W \oplus \underline{\mathbb{C}}^m$

For negative n, you could then use Bott periodicity:

Theorem 1.4. *(Bott periodicity)* $K^n(X) \cong K^{n+2}(X)$.

2 KO-theory

In the above definition, replace complex vector bundles with real vector bundles in the definition $\mathrm{Vect}(X)$, so $\mathrm{Vect}(X)$ is the set of isomorphism classes of real vector bundles on X, equipped with the operation $\oplus$.

Definition 2.1. *$KO^0(X)$ is the initial group receiving a map from* $\mathrm{Vect}(X)$ *which sends $\oplus$ to the group operation on $KO^0(X)$.*

The analogues of Proposition 1.2 and Corollary 1.3 hold as well. For $(V, W) \in KO^0(X)$, we can find $(V', \underline{\mathbb{R}}^n)$ such that $(V, W) = (V', \underline{\mathbb{R}}^n)$. Define

$$\mathrm{Th}(V, W) = \Sigma^{-n}\,\mathrm{Th}(V')$$

to be the object of the stable homotopy category, where $\mathrm{Th}(V')$ denotes the suspension spectrum of the Thom space of V'.

KO^0 also extends to a generalized cohomology theory. The real version of Bott periodicity is

Theorem 2.2. *(Bott periodicity) $\tilde{KO}^n(X) \cong \tilde{KO}^{n+8}(X)$.*

The periodicity isomorphism is induced by a product with a class in $\tilde{KO}^0(S^8) \cong \tilde{KO}^{-8}(S^0)$ constructed from Clifford algebras. The homotopy groups of KO are given $\tilde{KO}(S^n) \cong KO_n$ is

$$\mathbb{Z}, \mathbb{Z}/2, \mathbb{Z}/2, 0, \mathbb{Z}, 0, 0, 0, \mathbb{Z}$$

for

$$n = 0, 1, 2, 3, 4, 5, 6, 7, 8$$

respectively.

3 James Periodicity

By combining the Atiyah-Hirzebruch spectral sequence and K or KO-theory, we get a proof of a property of projective spaces called James periodicity. Let $\mathbb{L}$ denote the tautological bundle on real projective space. Recall the notation that

$$\mathbb{RP}_n^{m+n} = \mathrm{Th}(\mathbb{RP}^m, n\mathbb{L}).$$

We have seen that for n (and m) positive, we have $\mathbb{RP}_n^{m+n} = \mathbb{RP}^{m+n}/\mathbb{RP}^{n-1}$. From the above, we know that in the stable homotopy category, $\mathrm{Th}(\mathbb{RP}^m, n\mathbb{L})$ only depends on $n\mathbb{L}$ in $KO(\mathbb{RP}^m)$. We could equally well say that $\Sigma^{-n}\,\mathrm{Th}(\mathbb{RP}^m, n\mathbb{L})$ only depends on $n(\mathbb{L}-1)$ in $\tilde{KO}(\mathbb{RP}^m)$. Looking at the Atiyah-Hirzebruch spectral sequence, we have $\tilde{\mathrm{H}}^p(\mathbb{RP}^m, \tilde{KO}_{-q}) \Rightarrow \tilde{KO}^{p+q}(\mathbb{RP}^m)$. Along the $p + q = 0$ diagonal, we get $\mathbb{Z}/2$'s whenever there is a $\mathbb{Z}$ or a $\mathbb{Z}/2$ in $\tilde{KO}_{-q}$. For example, when $m = 8$, we have four $\mathbb{Z}/2$'s. This implies that $\tilde{KO}^0(\mathbb{RP}^m)$ has at most order 16. In particular, $16(\mathbb{L}-1)=0$, so we conlcude that $\Sigma^{-n}\operatorname{Th}(\mathbb{RP}^8, n\mathbb{L})$ only depends on n modulo 16. In other words

$$\Sigma^{-n}\mathbb{RP}_n^{8+n}$$

only depends on n modulo 16.

Exercise 3.1. *Do this for complex projective space, and other values of m.*

References

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