Lecture 13: Atiyah-Hirzebruch Spectral Sequence

2/17/15

Theorem 1.1. *Let A be a spectrum and let X be a finite-dimensional CW-complex. There exist spectral sequences*

$$E^2_{p,q} = \mathrm{H}_p(X; \pi_q A) \Rightarrow A_{p+q}X \qquad d^n : E^n_{p,q} \to E^n_{p-n,q+(n-1)}$$

$$E_2^{p,q} = \mathrm{H}^p(X; \pi_{-q} A) \Rightarrow A^{p+q}X \qquad d^n : E^n_{p,q} \to E^n_{p+n,q-(n-1)}.$$

Here we use the notation that for a spectrum A and a CW-complex X, $A_*X = \pi_*(\Sigma^\infty(X_+) \wedge A)$. When X is pointed, we have $\tilde{A}_*X = \pi_*(\Sigma^\infty X \wedge A)$, and if we replace the H's above with $\tilde{\mathrm{H}}$'s, we also get an Atiyah-Hirzebruch spectral sequence for the reduced generalized (co)homology of X.

Adam's says this theorem is probably due to Whitehead. See [A, p. 215]. With more work, one can make an Atiyah-Hirzebruch spectral sequence for X a spectrum which is bounded below in the sense that $\pi_q X = 0$ for all q sufficiently small.

Proof. We construct the first spectral sequence and leave the second as an exercise.

Let

$$\emptyset = X^{(-1)} \subset X^{(0)} \subset X^{(1)} \subset \ldots \subset\subset X^{(d)} = X$$

denote the skeletal filtration of X.

The long exact sequences

$$\ldots \to A_{p+q}X^{(p-1)} \xrightarrow{i_*} A_{p+q}X^{(p)} \xrightarrow{j} A_{p+q}(X^{(p)}/X^{(p-1)}) \xrightarrow{k} A_{p+q-1}X^{(p-1)} \to \ldots$$

can be assembled into the exact couple

$$\begin{array}{ccc} \oplus_{p,q} A_{p,q}X^{(p)} & \xrightarrow{\quad i_* \quad} & \oplus_{p,q} A_{p,q}X^{(p)} \\ {\scriptstyle k}\nwarrow & & \swarrow{\scriptstyle j} \\ & \oplus_{p,q} A_{p+q}(X^{(p)}/X^{(p-1)}) & \end{array}$$

We therefore have a spectral sequence. Let's work out the bidegree of the differential d^n on the nth page. As in last lecture, "$d^n = ji^{-(n-1)}k$." Set $E^1_{p,q} = A_{p+q}(X^{(p)}/X^{(p-1)})$. Then $i^{-(n-1)}$ moves back $n-1$ in the p coordinate, and k moves back another 1 in the p coordinate. Thus

$$d^n : \text{subquotient of } A_{p+q}(X^{(p)}/X^{(p-1)}) \to \text{subquotient of } A_{p+q-1}(X^{(p-n)}/X^{(p-n-1)}).$$

Thus $d^n : E^n_{p,q} \to E^n_{p-n,q+n-1}$, as claimed.

We now identify $E^2_{p,q}$ with $\mathrm{H}_p(X;\pi_q A)$. We have

$$E^1_{p,q} \cong A_{p+q}(X^{(p)}/X^{(p-1)}) \cong A_{p+q} \vee_{C_p} S^p \cong \oplus_{C_p}\pi_q A,$$

where C_p denotes the set of p-cells of X. This is the group of p-cellular chains with coefficients in $\pi_q A$, so it remains to identify $d^1 : \oplus_{C_p}\pi_q A \to \oplus_{C_{p-1}}\pi_q A$ with the cellular boundary map.

Choose a p-cell α and a $(p-1)$-cell β. We need to see that the composite

$$\pi_q A \cong \pi_{p+q}(A \wedge S^p) \stackrel{\alpha_*}{\to} \pi_{p+q} \vee_{C_p} (A \wedge S^p) \stackrel{d^1}{\to} \oplus_{C_{p-1}}\pi_q A \stackrel{p_\beta}{\to} \pi_q A$$

is multiplication by the degree of $S^{p-1} \stackrel{\alpha}{\to} X^{(p-1)} \to X^{(p-1)}/X^{(p-2)} \cong \vee_{C_{p-1}} S^{p-1} \stackrel{p_\beta}{\to} S^{(p-1)}$. Unwrapping definitions, we have that d^1 is the composite of

$$\pi_{p+q}A \wedge (X^{(p)}/X^{(p-1)}) \to \pi_{p+q}A \wedge X^{(p-1)} \to \pi_{p+q}A \wedge X^{(p-1)}/X^{(p-2)}.$$

Thus the claim is that a degree n map $S^{p-1} \to S^{p-1}$ induces multiplication by n on A_*. By homotopy invariance of the generalized homology theory A_*, we may check this for the degree n map given by

$$S^{p-1} \to \vee_{i=1}^n S^{p-1} \stackrel{\vee \mathrm{id}}{\to} S^{p-1},$$

where the first map is the pinch map. The claim then follows from the fact that generalized homology theories turn wedge sums into direct sums, and the identity induces an identity map. (This argument was already written out on page 4 of Lecture 11.)

It remains to prove the convergence to $A_{p+q}X$. Since $E^1_{p,q}X = A_{p+q}(X^{(p)}, X^{(p-1)})$, $E^1_{p,q}$ is only nonzero when $0 \le p \le \dim X$. Since $E^n_{p,q}$ is a sub quotient of $E^1_{p,q}$, the same statement holds for $E^n_{p,q}$. Since d^n decreases the p coordinate by n, for $n > d+1$, we have that $d^n = 0$. Thus for all $n > d+1$, we have $E^{d+1}_{p,q} \cong E^n_{p,q}$. So $E^\infty_{p,q}$ is well-defined; it is $E^{d+1}_{p,q}$.

$A_{p+q}X$ is filtered by $\mathcal{F}_p = \mathrm{Image}(A_{p+q}X^{(p)} \to A_{p+q}X)$. We claim that $\mathrm{gr}\, A_n X \cong \oplus_{p+q=n} E^\infty_{p,q}$. In fact, we claim that $\mathcal{F}_p/\mathcal{F}_{p-1} \cong E^\infty_{p,n-p}$. On the Nth page, any element $\overline{e}$ of $E^{N+1}_{p,q}$ must be the image of an element $e \in E^1_{p,q} \cong$

$A_{p+q}(X^{(p)}, X^{(p-1)})$ and in the kernel of "$d^N = ji^{-N}k$." This means that the boundary ∂e must be 0 in $A_{p+q-1}(X^{(p-1)}, X^{(p-N-1)})$. This implies that e determines an element of $A_{p+q}(X^{(p)}, X^{(p-N-1)})$. Furthermore, $\bar{e}$ is only determined modulo the images of the the d^i for $i < N$. By the same reasoning, the images of these d^i are boundaries from $A_{p+q+1}(X^{(p+i)}, X^{(p)})$. Thus

$$E^{N+1}_{p,q} \cong \frac{\text{Image}(A_{p+q}(X^{(p)}, X^{(p-N-1)}) \to A_{p+q}(X^{(p)}, X^{(p-1)})}{\text{Image}(A_{p+q+1}(X^{(p+N)}, X^{(p)}) \to A_{p+q}(X^{(p)}, X^{(p-1)}))}$$

Taking N sufficiently large we have

$$E^{N+1}_{p,q} \cong \frac{\text{Image}(A_{p+q}(X^{(p)}) \to A_{p+q}(X^{(p)}, X^{(p-1)}))}{\text{Image}(A_{p+q+1}(X, X^{(p)}) \to A_{p+q}(X^{(p)}, X^{(p-1)}))}$$

By arguments with the LES of a triple, we get

$$\begin{aligned} E^{N+1}_{p,q} &\cong \frac{\text{Image}(A_{p+q}(X^{(p)}) \to A_{p+q}(X^{(p)}, X^{(p-1)}))}{\text{Ker}(A_{p+q}(X^{(p)}, X^{(p-1)}) \to A_{p+q}(X, X^{(p-1)}))} \\ &\cong \frac{\text{Image}(A_{p+q}(X^{(p)}) \to A_{p+q}(X^{(p)}, X^{(p-1)}))}{\text{Ker}(A_{p+q}(X^{(p)}, X^{(p-1)}) \to A_{p+q}(X, X^{(p-1)}))} \\ &\cong \frac{\mathcal{F}_p}{\mathcal{F}_{p-1}} \end{aligned}$$

□

Exercise 1.2. *a) The last isomorphism takes a little diagram chasing. See if you can complete those steps.*

b)More ambitious exercise: give the proof of the other spectral sequence in the theorem.

We're going to talk about topological complex K-theory next. For now, let's assume we know that there is a spectrum K whose even indexed spaces are $\mathbb{Z} \times BU$ and whose odd indexed spaces are U. Furthermore $\pi_* K = \mathbb{Z}$ if $*$ is even and $\pi_* K = 0$ if $*$ is odd.

Example 1.3. *Apply the Atiyah-Hirzebruch spectral sequence to $X = \mathbb{CP}^n$ and $A = K$. The E^2 page $E^2_{p,q} = H^p(\mathbb{CP}^n, \pi_{-q}K)$ is a checkerboard pattern in the right half plane with $\mathbb{Z}$'s on the squares with both coordinates even and such that $0 \le p \le 2n$. Since the differentials always change the parity of one of the coordinates, ALL the differentials are automatically 0. We conclude*

$$K^* \mathbb{CP}^n = \begin{cases} \oplus_{i=0}^n \mathbb{Z} & \textit{if } * \textit{ even} \\ 0 & \textit{otherwise} \end{cases}.$$

Exercise 1.4. *Compute $\tilde{K}^*(\mathbb{RP}^{2n})$ as an additive group.*

References

[A] J.F. Adams, *Stable Homotopy and Generalized Homology* Chicago Lectures in Mathematics, The University of Chicago Press, 1974.