Lecture 12: Spectral sequences

2/15/15

1 Definition

A *differential* group (E, d) (respectively algebra, module, vector space etc.) is a group (respectively algebra, module, vector space etc.) E together with a morphism $d : E \to E$ such that $d^2 = 0$. The homology $\mathrm{H}(E, d)$ of (E, d) is defined to be the group

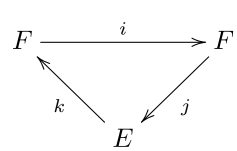
$$\mathrm{H}(E, d) = \operatorname{Ker} d / \operatorname{Image} d.$$

Definition 1.1. *A* spectral sequence *is a sequence of differential groups* (E_n, d_n) *for* $n \geq 2$ *(or* 1 *or* 0*) such that* $E_n \cong \mathrm{H}(E_{n-1}, d_{n-1})$.

This definition is useful when E_2 can be computed and the E_n's stabilize to some E_∞ that we wish to compute.

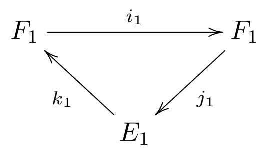
2 Spectral sequence associated to an exact couple

An *exact couple* is a pair F, E of groups and a diagram



which is exact. Define $d = jk : E \to E$. Then, by exactness, $d^2 = j(kj)k = 0$ because $kj = 0$, i.e., (E, d) is a differential group.

Given an exact couple, we can form another exact couple called the *derived couple*:



(1)

with the following definitions. $F_1 = \text{Image}(i)$, $E_1 = \text{H}(E, d)$, $i_1 = i|_{F_1}$, $k_1 = k$, and "$j_1 = ji^{-1}$." Since i is not invertible, the definition $j_1 = ji^{-1}$ really means that given $f \in F_1 = i(F)$, we can choose f' such that $if' = f$. Then $j_1 f = jf'$. In order for the definitions of j_1 and k_1 to make sense, we need to check that jf' does not depend on the choice of f', and k is a well-defined function on $\text{Ker}(jk)/\text{Image}(jk)$ whose image is in the image of i. Note that jf' can be modified by j applied to an element in the kernel of i. This kernel is the image of k by construction. Thus, jf' can only be modified by an element in $\text{Image}(jk)$, showing that j_1 is well-defined. The others can also be checked in an entirely straight-forward manner.

Exercise 2.1. *Show* (1) *does indeed define an exact couple, i.e., the diagram is exact.*

If you are not familiar with spectral sequences, it is a good idea to think a few of these things through.

Repeating this process, we obtain a spectral sequence (E_n, d_n).

We can write down the terms of this spectral sequence explicitly. For example, $F_n \subseteq F$ is the image of i^n and i_n is the restriction of i to F_n. $d_n = j_n k_n$ is given by the formula "$d_n = ji^{-n}k$." Adams says that these explicit formulas probably come from a point of view due to Eilenberg, whereas the exact couple point of view is due to Massey. There is a short description of the explicit formula method in [L].

3 Convergence

A first approximation to what "convergence" means is that a spectral sequence converges to a group A if $E_n = A$ for sufficiently large n. But this is not sufficiently general to be useful.

A group (respectively algebra, module, vector space etc.) A is said to be *filtered* by $n \in \mathbb{Z}$ if there is a sequence of subgroups

$$\ldots \subset \mathcal{F}_n A \subset \mathcal{F}_{n+1} A \subset \mathcal{F}_{n+2} A \subset \ldots$$

of A. We will assume that $\cup_n \mathcal{F}_n A = A$, and that $\cap_n \mathcal{F}_n A = 0$.

A group (respectively algebra, module, vector space etc.) A is said to be $\mathbb{Z}$-*graded* or just *graded* (respectively *bigraded*) if $A = \oplus_{n \in \mathbb{Z}} A_n$ (respectively $A = \oplus_{p,q \in \mathbb{Z}} A_{p,q}$). Call A_p (resp. $A_{p,q}$) the pth (respectively (p,q)th) homogenous piece.

The *associated graded* of a filtered group A is $\mathrm{gr}\, A = \oplus_n \mathcal{F}_{n+1} A / \mathcal{F}_n A$.

Frequently spectral sequences have additional structure such that (E_n, d_n) are graded (respectively bigraded). Suppose that A is a filtered group (respectively filtered and graded). We say that (E_n, d_n) *converges* to A

$$(E_n, d_n) \Rightarrow A$$

if for each p (respectively (p,q)), the homogenous pieces of the E_n stabilize to the associated graded of A. Note that when A has a grading as well as a filtration, the associated graded is bigraded, so that's consistent with having a bigrading on the spectral sequence. Degree considerations and indexing of spectral sequences can certainly be messy. On the other hand, spectral sequences turn out to be very powerful.

4 Spectral sequence associated to a double complex.

A *double complex* will be a bunch of groups (or modules etc.) $A^{p,q}$ for $\mathbb{Z} \ni p, q \geq 0$ and differentials $d : A^{p,q} \to A^{p-1,q}$, $d' : A^{p,q} \to A^{p,q-1}$ such that the differentials either commute, meaning $dd' = d'd$, or anti-commute, meaning $dd' + d'd = 0$. Both come up, and you can go from one to the other by changing d' to $(-1)^p d'$. Let's assume we are in the anti-commuting case.

Define $\mathrm{Tot}\, A^{*,*}$ to be the differential graded group whose nth homogenous piece is $\oplus_{p+q=n} A^{p,q}$ and whose differential is $D = d + d'$.

There are two natural spectral sequences associated to this double complex, and both converge to $\mathrm{H}(\mathrm{Tot}\, A^{*,*}, D)$. Here is one.

Note that there is a filtration $\mathcal{F}_n = \mathcal{F}_n \,\mathrm{Tot}\, A^{*,*} = \oplus_{p \leq n} A^{p,q}$ on $\mathrm{Tot}\, A^{*,*}$ such that $D : \mathcal{F}_n \to \mathcal{F}_n$. We have a short exact sequence of differential objects

$$0 \to \mathcal{F}_n \to \mathcal{F}_{n+1} \to \mathcal{F}_{n+1}/\mathcal{F}_n \to 0$$

which gives rise to a long exact sequence in homology with respect to D

$$\ldots \to \mathrm{H}_i\, \mathcal{F}_n \to \mathrm{H}_i\, \mathcal{F}_{n+1} \to \mathrm{H}_i\, \mathcal{F}_{n+1}/\mathcal{F}_n \to \mathrm{H}_{i-1}\, \mathcal{F}_n \to \ldots \tag{2}$$

Define

$$F_1 = \oplus_{p,q}\, \mathrm{H}_{p+q}\, \mathcal{F}_p$$

$$E_1 = \oplus_{p,q}\, \mathrm{H}_{p+q}(\mathcal{F}_p/\mathcal{F}_{p-1}).$$

Define i_1,j_1, and k_1 using the maps from the long exact sequence, meaning define $i_1 : F \to F$ by the maps $\mathrm{H}_{p+q}\, \mathcal{F}_p \to \mathrm{H}_{p+q}\, \mathcal{F}_{p+1}$ induced from the inclusions; define $j_1 : F \to E$ by the maps $\mathrm{H}_{p+q}\, \mathcal{F}_{p+1} \to \mathrm{H}_{p+q}\, \mathcal{F}_{p+1}/\mathcal{F}_p$ induced from the quotient; define $k_1 : E \to F$ using the boundary maps. This is an exact couple because (2) is exact. Therefore we have constructed a spectral sequence.

The other spectral sequence is obtained by switching the roles of p and q in the definition of the filtration.

To use this spectral sequence and to see its convergence, it is useful to draw it. Place every $A^{p,q}$ on the (p,q) spot in the first quadrant. There are arrows d' forming vertical lines pointing down. Replace each $A^{p,q}$ by $A_1^{p,q} = \operatorname{Ker} d'/\operatorname{Image} d'$ the homology with respect to d'. Note that $D = d'$ on all $\mathcal{F}_{n+1}/\mathcal{F}_n$. Thus $E_1 = \oplus_{p,q} A_1^{p,q}$ is the direct sum of all the groups in our first quadrant. Moreover, the d_1 from our spectral sequence can be identified with d'. Furthermore, the differential d_n is of bidegree $(-n, n-1)$. (Exercise.) In particular, for any (p,q) eventually all of the differentials leaving or entering the group at the (p,q)th spot are 0 so the spectral sequence converges. Call $E_\infty^{(}p,q)$ the limiting group in the (p,q)th spot.

The filtration on $\operatorname{Tot} A^{*,*}$ induces a filtration on $\mathrm{H}_n \operatorname{Tot} A^{*,*}$, by defining $\mathcal{F}_p\, \mathrm{H}_n \operatorname{Tot} A^{*,*}$ to be the image of the map on H_n induced by $\mathcal{F}_p \operatorname{Tot} A^{*,*} \to \operatorname{Tot} A^{*,*}$.

Exercise 4.1. *Show that* $E_\infty^{(p,q)} = \mathcal{F}_p\, \mathrm{H}_{p+q} \operatorname{Tot} A^{*,*}/\mathcal{F}_{p-1}\, \mathrm{H}_{p+q} \operatorname{Tot} A^{*,*}$. *In other words, the limiting groups can be identified with the associated graded of the homology of* $\operatorname{Tot} A^{*,*}$.

Example 4.2. *Given the commutative diagram with exact rows*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & 0\ , \\ & & {\scriptstyle f}\downarrow & & {\scriptstyle g}\downarrow & & {\scriptstyle h}\downarrow & & \\ 0 & \longrightarrow & A' & \longrightarrow & B' & \longrightarrow & C' & \longrightarrow & 0 \end{array}$$

it is a standard exercise to show that there is an exact sequence

$$0 \to \operatorname{Ker} f \to \operatorname{Ker} g \to \operatorname{Ker} h \to \operatorname{coKer} f \to \operatorname{coKer} g \to \operatorname{coKer} h \to 0.$$

This follows from the two spectral sequences just constructed. Namely, we can view this diagram as a double complex. Since the rows are exact, one of our spectral sequences converges to 0*, so it follows that* Tot *has* 0 *homology. Thus the other spectral sequence must also converge to* 0*. This happens if and only if the claimed sequence is exact.*

References

[A] J.F. Adams, *Stable Homotopy and Generalized Homology* Chicago Lectures in Mathematics, The University of Chicago Press, 1974.

[L] Serge Lang, *Algebra*, Graduate Texts in Mathematics.

[MT] Robert Mosher and Martin Tangora, *Cohomology Operations and Applications in Homotopy Theory*, Dover, 1968, 2008.

[V] Ravi Vakil, *Course notes for Math 216.*