Lecture 11: Generalized homology

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We work in the stable homotopy category. For X and Y objects of the stable homotopy category, we are assuming the existence of a commutative, associative smash product with unit such that $X \wedge (-)$ preserves cofiber sequences and wedge sums. We also assume the existence for each X, Y of a spectrum $F(X,Y)$ such that

$$[W \wedge X, Y] = [W, F(X,Y)]. \tag{1}$$

1.1 Group structure

For any spectra X and Y, and integer r, there is a natural abelian group structure on $[X,Y]_r$: for $f, g \in [X,Y]_r = [\Sigma\Sigma^{-1}X, Y]_r$, we have $f \vee g \in [X \vee X, Y]_r$ and the pinch map $p : \Sigma\Sigma^{-1}X \to \Sigma\Sigma^{-1}X \vee \Sigma\Sigma^{-1}X$. Define $f + g = (f \vee g) \circ p$. The standard arguments for π_1 being a group show that $[X,Y]_r$ is indeed a group. For example, the sum of a map and its inverse is the constant map because the map

$$\Sigma\Sigma^{-1}X \to \Sigma\Sigma^{-1}X \vee \Sigma\Sigma^{-1}X \overset{1\vee(-1)}{\to} \Sigma\Sigma^{-1}X$$

is null-homotopic. Since X is also isomorphic to $\Sigma^2(\Sigma^{-2}X)$, we could pinch along the other suspension coordinate. One might worry that this would give a different operation. In fact, these operations are the same on the level of $[X,Y]_r$ as can be seen by the Eckmann-Hilton argument:

Proposition 1.1. *Suppose X is a set equipped with two binary operations $+$ and $*$ such that each operation has an identity and for all a,b,c, and d in X*

$$(a + b) * (c + d) = (a * c) + (b * d).$$

The $+$ and $*$ are equal and both commutative and associative.

The proof is "the same" as the standard proof that π_2 is commutative as in the picture [H, p. 340]. You could do it as an exercise, or it is available on Wikipedia.

1.2 Generalized homology

Definition 1.2. *Let E be a spectrum. The generalized E-homology of degree r is the functor from the stable homotopy category to abelian groups*

$$X \mapsto E_*(X) = \pi_*(E \wedge X).$$

Definition 1.3. *A* generalized reduced homology theory *on based CW-complexes is a sequence of functors $\tilde{h}_* : \mathrm{ho}\,\mathbf{CW}_* \to \mathbf{Ab}$ from the homotopy category of CW-complexes equipped with a base point to abelian groups for $* \in \mathbb{Z}$ such that*

1. *There are natural boundary maps $\tilde{h}_*(X/Y) \to \tilde{h}_{*-1}(Y)$ such that*

   $$\ldots \to \tilde{h}_*(Y) \to \tilde{h}_*(X) \to \tilde{h}_*(X/Y) \to \tilde{h}_{*-1}(A) \to \ldots$$

   is exact for each CW pair (X, Y).

2. *The map $\oplus_{a\in A}\tilde{h}_*(X_a) \to \tilde{h}_*(\vee_{a\in A} X_a)$ is an isomorphism.*

Remark 1.4. *A generalized reduced homology theory $\tilde{h}_*$ on based CW-complexes can be extended to a functor on non-based CW-complexes by setting $\tilde{h}_*(X) = \mathrm{Ker}(\tilde{h}_*(X_+) \to \tilde{h}_*(S^0))$. A choice of base point of a CW complex X gives a map $S^0 \to X_+$, which in turn produces a direct sum decomposition $\tilde{h}_*(X_+) \cong \tilde{h}_*(X) \oplus \tilde{h}_*(S^0)$. Thus for a based CW complex, the new definition is canonically isomorphic to the old.*

Proposition 1.5. *The functors $\mathrm{ho}\,\mathbf{CW}_* \to \mathbf{Ab}$ defined $X \mapsto E_*(\Sigma^\infty X)$ defines a generalized reduced homology theory for any object E of the stable homotopy category.*

Proof. For a CW-pair, there is a homotopy equivalence $X \cup CY \cong X/Y$, whence

$$Y \to X \to X/Y$$

is a cofiber sequence. Since $E \wedge (-)$ preserves cofiber sequences, we have a cofiber sequence

$$E \wedge Y \to E \wedge X \to E \wedge (X/Y).$$

The long exact sequence in homotopy groups associated to a cofiber sequence gives 1.

When A is finite, 2 follows from the fact that the long exact sequence in π_* associated to the cofiber sequence

$$E \wedge X \to (E \wedge X) \vee (E \wedge Y) \to (E \wedge Y)$$

splits into short exact sequences using the map $(E \wedge Y) \to (E \wedge X) \vee (E \wedge Y)$.

In general, we have $E \wedge (\vee_{a\in A} X_a) \cong \vee_{a\in A}(E \wedge X_a)$. Then $E_*(\vee_{a\in A} X_a) \cong \pi_*(\vee_{a\in A}(E\wedge X_a))$. Recall that $\pi_*(\vee_{a\in A}(E\wedge X_a)) \cong \mathrm{colim}_{n\to\infty}\, \pi_{n+*}(\vee_{a\in A}(E \wedge X_a)_n)$. Since S^{n+*} is compact, any element of $\pi_{n+*}(\vee_{a\in A}(E\wedge X_a)_n)$ is in the image of a $\pi_{n+*}(\vee_{a\in A'}(E\wedge X_a)_n)$ with A' finite. Since $S^{n+*}\times[0,1]$ is also compact, any homotopy between two maps $S^{n+*}\to \vee_{a\in A}(E\wedge X_a)_n$ also factors through some $(\vee_{a\in A'}(E\wedge X_a)_n)$ with A' finite. Thus $\pi_{n+*}(\vee_{a\in A}(E\wedge X_a)_n) \cong \mathrm{colim}_{A'\subset A}\, \pi_{n+*}(\vee_{a\in A'}(E\wedge X_a)_n)$ where the colimit is taken over finite subsets A' of A. Since colimits commute, we have that

$$E_*(\vee_{a\in A} X_a) \cong \operatorname*{colim}_{A'\subset A}\operatorname*{colim}_{n\to\infty} \pi_{n+*}(\vee_{a\in A'}(E\wedge X_a)_n) \cong \operatorname*{colim}_{A'\subset A} \oplus_{A'} E_*(X_a) \cong \oplus_A E_*(X_a).$$

□

The converse to Proposition 1.5 holds as well, but we will not prove it. In fact, Adams showed something better:

Theorem 1.6. *([A2]) Let E be a spectrum and let $\tilde{h}_*$ be a generalized reduced homology theory. Suppose we have a map of homology theories $f : E_* \to \tilde{h}_*$. Then there exists a spectrum F together with an isomorphism $F_* \cong \tilde{h}_*$ and a map $E \to F$ such that the induced map $E_* \to F_* \cong \tilde{h}_*$ is f.*

Let A be an abelian group and HA denote the Eilenberg-MacLane spectrum associated to A.

Proposition 1.7. *There is a natural isomorphism between the singular reduced homology with coefficients in A and the generalized homology of HA,*

$$HA_*X \cong \tilde{\mathrm{H}}_*(X, A).$$

On the left hand side, we may choose any 0 simplex of X as the base point to take the suspension spectrum.

Proof. Let X be a pointed CW-complex with n-skeleton $X^{(n)}$. Let C_n be the set of n-cells of X, i.e. $X^{(n)}/X^{(n-1)} \cong \vee_{C_n} S^n$. We show that HA_*X can be computed with the reduced cellular chain complex associated to A.

Note that $HA_*(S^n) \cong HA_{*-n} \cong \begin{cases} 0 & \text{if } * \neq n \\ \mathbb{Z} & \text{if } * = n. \end{cases}$. Thus by Proposition 1.5,

$$HA_*(X^{(n)}/X^{(n-1)}) \cong \begin{cases} \oplus_{C_n} A & \text{if } * = n \\ 0 & \text{if } * \neq n. \end{cases}.$$

Considering the cofiber sequences

$$X^{(n-1)} \to X^{(n)} \to X^{(n)}/X^{(n-1)}$$

and the associated long exact sequences in HA_* (Proposition 1.5), we have that $\operatorname{colim}_{m\to\infty} HA_n X^{(m)} \cong HA_n X^{(n+1)}$. Since S^n is compact, we can show similarly to the argument above that $\operatorname{colim}_{m\to\infty} HA_n X^{(m)} \cong HA_n X$. It follows that $HA_* X$ can be computed with the chain complex

$$\ldots \to HA_n(X^{(n)}/X^{(n-1)}) \to HA_{n-1}(X^{(n-1)}/X^{(n-2)}) \to \ldots$$

where the differential is the composite of the boundary map $HA_n(X^{(n)}/X^{(n-1)}) \to HA_{n-1}X^{(n-1)}$ with the map induced from the quotient map $X^{(n-1)} \to X^{(n-1)}/X^{(n-2)}$.

To identify this complex with the cellular complex computing $\tilde{\mathrm{H}}_*(X, A)$ it remains to show that the degree m map $S^n \to S^n$ induces multiplication by $m : A \to A$ on HA_n. We may show this for the degree m map given as an m-fold pinch map followed by $\vee_{i=1}^m i$

$$S^n \to \vee_{i=1}^m S^n \overset{\vee_{i=1}^m 1}{\to} S^n.$$

The result follows by applying HA_n and using the isomorphism $HA_n(\vee_{i=1}^m S^n) \cong \oplus_{i=1}^m A$ and the fact that 1 induces the identity.

□

References

[A1] J.F. Adams, *Stable Homotopy and Generalized Homology* Chicago Lectures in Mathematics, The University of Chicago Press, 1974.

[A2] J.F. Adams, *A variant of E.H. Brown's Representability Theorem*, Topology, vol 10, pp 185-198, 1971.

[H] Allen Hatcher, *Algebraic Topology.*