Lecture 10: Atiyah Duality

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Let X be a finite CW complex. We saw last time that the Spanier-Whitehead dual of X is equivalent to $\Sigma^{-(n-1)}(S^n - X)$ where $X \to S^n$ is any non-surjective embedding of X into a sphere, and such an embedding always exists.

Such an embedding gives an embedding of X in $\mathbb{R}^n$ by removing a point of S^n. Adding this point to X, we have the formula

$$\mathbb{D}(X_+) \cong \Sigma^{-(n-1)}(\mathbb{R}^n - X). \tag{1}$$

We may also replace $\mathbb{R}^n$ by the homeomorphic open disk D^n.

Note that

$$(\mathbb{R}^n - X) \to \mathbb{R}^n \to \Sigma(\mathbb{R}^n - X)$$

is a cofiber sequence, because $\mathbb{R}^n$ is contractible. Thus we may rewrite (1) as $\mathbb{D}(X_+) \cong \Sigma^{-n}(\mathbb{R}^n/(\mathbb{R}^n - X))$. For any neighborhood N of X in $\mathbb{R}^n$, we thus also have $\mathbb{D}(X_+) \cong \Sigma^{-n}N/(N - X)$

A sufficiently small neighborhood N deformation retracts back to X. Furthermore, if X is a manifold, N can be chosen so as to identify with the disk bundle $D(N_X\mathbb{R}^n)$ of the normal bundle $N_X\mathbb{R}^n \to X$ with ∂N identified with the sphere bundle $S(N_X\mathbb{R}^n)$. Such an N is a *tubular neighborhood*. We then have $N/(N - X) \cong N/\partial N \cong D(N_X\mathbb{R}^n)/S(N_X\mathbb{R}^n) \cong \mathrm{Th}(N_X\mathbb{R}^n)$. Combining with the previous we obtain

$$\mathbb{D}(X_+) \cong \Sigma^{-n}\,\mathrm{Th}(N_X\mathbb{R}^n) \tag{2}$$

Recall that if we add a trivial bundle $\underline{\mathbb{R}} \to X$ to a vector bundle $V \to X$, the resulting thom space $\mathrm{Th}(V \oplus \underline{\mathbb{R}}) \cong \Sigma\,\mathrm{Th}(V)$. This leads to the definition $\mathrm{Th}(V \ominus \mathbb{R}^n) \cong \Sigma^n\,\mathrm{Th}(V)$ for n positive or negative. Note that the Whitney sum of the tangent bundle and the normal bundle of $X \to \mathbb{R}^n$ is the trivial bundle of rank n, i.e. $N_X\mathbb{R}^n \oplus T_X \cong \underline{R}^n$. Rearranging terms, we have:

Theorem 1.1. *If X is a compact manifold without boundary, $\mathbb{D}(X_+) \cong \mathrm{Th}(-T_X)$.*

This implies Poincaré duality [H, Theorem 3.30], but to see this, we need to take as given the Thom isomorphism theorem, which says that if $V \to X$ is an orientable vector bundle of rank n, then there is a natural isomorphism $\tilde{\mathrm{H}}^{*+n}(\mathrm{Th}(V), \mathbb{Z}) \cong \mathrm{H}^*(X, \mathbb{Z})$. One way to think about this is that orientable is a condition which says that V behaves like a trivial bundle after taking cohomology, so $\tilde{\mathrm{H}}^*(\mathrm{Th}(V), \mathbb{Z}) \cong \tilde{\mathrm{H}}^*(\mathrm{Th}(\underline{\mathbb{R}^n}), \mathbb{Z})$. Since $\mathrm{Th}(\underline{\mathbb{R}^n}) \cong S^n \wedge (X_+)$, we have $\tilde{\mathrm{H}}^*(\mathrm{Th}(V), \mathbb{Z}) \cong \tilde{\mathrm{H}}^*(\Sigma^n X_+, \mathbb{Z}) \cong \mathrm{H}^{*-n}(X, \mathbb{Z})$. Furthermore, for any abelian group A, there is a notion of A-orientable, and a natural isomorphism $\tilde{\mathrm{H}}^{*+n}(\mathrm{Th}(V), A) \cong \mathrm{H}^*(X, A)$ when V is A-oriented. Furthermore, for E a spectrum, there is a notion of E-orientable, and a natural isomorphism $E^{*+n}\,\mathrm{Th}(V) \cong E^*(X_+)$ when V is E-oriented. We'll discuss this later.

Corollary 1.2. *(Poincaré duality) Suppose that X is a compact n-manifold without boundary and E is a spectrum such that the tangent space T_X of X is E-orientable. Then*

$$E_*(X_+) \cong E^{n-*}(X_+).$$

In particular, if X is orientable, then $\mathrm{H}_*(X, \mathbb{Z}) \cong \mathrm{H}^{n-*}(X, \mathbb{Z})$.

Proof. Since $\mathbb{D}(X_+) \cong \mathrm{Th}(-T_X)$, we have that $[\mathbb{D}(X_+), E]_* \cong [\mathrm{Th}(-T_X), E]_*$. The right hand side is $E^{-*}\,\mathrm{Th}(-T_X)$. By the Thom isomorphism theorem,

$$E^{-*}\,\mathrm{Th}(-T_X) \cong E^{-*-\mathrm{rank}(-T_X)}(X_+) \cong E^{n-*}X_+.$$

The left hand side is $[\mathbb{D}(X_+), E]_* \cong [S^0, (X_+) \wedge E]_* \cong E_*(X_+)$. □

Via the formula $\mathrm{Th}(\underline{R}^n) \cong S^n \wedge X_+$, X_+ can be interpreted as the Thom space of the trivial 0-dimensional bundle. We can generalize Theorem 1.1 with a formula, due to Atiyah [At], for the dual of any Thom space over X. For a compact n-manifold with boundary, view the tangent bundle as a rank n vector bundle together with a distinguished sub-bundle of rank $n-1$ when restricted to the boundary.

Theorem 1.3. *(Atiyah duality) If X is a compact manifold with boundary ∂X then*

$$\mathbb{D}(X/\partial X) \cong \mathrm{Th}(-T_X).$$

If X is a compact manifold without boundary and V is a smooth vector bundle, then

$$D(\mathrm{Th}\,V) \cong \mathrm{Th}(-T_X - V).$$

Proof. Embed X into the Euclidean n-disk D^n, so that ∂X is embedded into $\partial D^n = S^{n-1}$. Assume these embeddings are cellular and that X is transverse to S^{n-1}. Choose a tubular neighborhood N of X such that $N \cap S^{n-1}$ is a tubular neighborhood of ∂X.

$X/\partial X$ is homotopy equivalent to $Y = X \cup \mathcal{C}\partial X \subset D^n \cup \mathcal{C}S^{n-1} \cong S^n$, where as before $\mathcal{C}$ denotes the (unreduced) cone. Applying Alexander duality to $Y \subset S^n$, we have

$$\mathbb{D}(X/\partial X) \cong \Sigma^{-(n-1)}(S^n - Y).$$

Note that $\mathcal{N} = N \cup \mathcal{C}(N \cap S^{n-1})$ is a neighborhood of Y, which deformation retracts onto Y. Thus $S^n - Y \cong (S^n - \mathcal{N})$. Since the cone point is not in $(S^n - \mathcal{N})$, we have that $(S^n - \mathcal{N}) \cong D^n - N$.

Thus

$$\begin{aligned}
\mathbb{D}(X/\partial X) &\cong \Sigma^{-(n-1)}(S^n - Y) \\
&\cong \Sigma^{-n}\Sigma(S^n - Y) \\
&\cong \Sigma^{-n}\Sigma(S^n - \mathcal{N}) \\
&\cong \Sigma^{-n}\Sigma(D^n - N) \\
&\cong \Sigma^{-n}\,\mathrm{Th}(N_X D^n) \\
&\cong \mathrm{Th}(-T_X)
\end{aligned}$$

The second assertion follows from the first. Let $D(V)$ and $S(V)$ denote the disk and sphere bundles of V respectively. Then $D(V)$ is a compact manifold with boundary $S(V)$, so $\mathbb{D}\,\mathrm{Th}(V) \cong \mathrm{Th}(-T_{D(V)})$. The kernel of $T_V \to T_X$ is the fiber-wise tangent bundle, which is V. Let $v : D(V) \to X$ denote the restriction of $V \to X$ to $D(V)$. Then there is a short exact sequence $v^*V \to T_{D(V)} \to v^*T_X$. It turns out that up to homotopy the Thom space of a vector bundle only depends on the bundle's class in K-theory. Thus $\mathrm{Th}(-T_{D(V)}) \cong \mathrm{Th}(-v^*(V \oplus T_X))$. Since v is a homotopy equivalence, $\mathrm{Th}(-v^*(V \oplus T_X)) \cong \mathrm{Th}(-(V \oplus T_X))$, as claimed. □

Example 1.4. Truncated projective space *Recall that $X = \mathbb{RP}^n = S^n/(v \sim -v)$ and that the tautological line bundle $\mathbb{L} \to \mathbb{RP}^n$ is the vector bundle whose fiber over $v \in S^n$ is $\ell = \mathbb{R}v$.*

Put the standard inner product on $\mathbb{R}^{n+1}$, and let $\ell^\perp$ donate the n-dimensional subspace of $\mathbb{R}^{n+1}$ of those vectors perpendicular to ℓ. Then $T_{[v]}\mathbb{RP}^n \cong \mathrm{Hom}(\ell, \ell^\perp)$. Since $\ell \oplus \ell^\perp \cong \mathbb{R}^{n+1}$, we have

$$0 \to \mathbb{R} = \mathrm{Hom}(\ell, \ell) \to \mathrm{Hom}(\ell, \mathbb{R}^n) \to \mathrm{Hom}(\ell, \ell^\perp) \to 0.$$

This globalizes to an exact sequence

$$0 \to \underline{R} \to (\mathbb{L}^*)^{n+1} \to T_{\mathbb{RP}^n} \to 0$$

of vector bundles on $\mathbb{RP}^n$.

It turns out that up to homotopy the Thom space of V only depends on the class of V in K-theory. Then it follows that

$$\mathbb{D}\mathbb{R}\mathbb{P}^n \cong \mathrm{Th}(-T_{\mathbb{R}\mathbb{P}^n}) \cong \mathrm{Th}(\underline{R} - (\mathbb{L}^*)^{n+1}).$$

We can identify $\mathrm{Th}(\mathbb{L}^*)^k$ with a truncated projective space for any k as follows. Any linear function from $\ell \in \mathbb{R}^{n+1}$ to $\mathbb{R}^k$ determines a graph in $\mathbb{R}^k \times \mathbb{R}^{n+1}$, which in turn determines an element of $(\mathbb{R}^k \times \mathbb{R}^{n+1} - \{0\})/\mathbb{R}^* \cong \mathbb{R}\mathbb{P}^n + k$. The only elements of $\mathbb{R}\mathbb{P}^n + k$ not equal to such a graph, are lines in $\mathbb{R}^k$. Thus we have a vector bundle

$$V = \mathbb{R}\mathbb{P}^{n+k} - \mathbb{R}\mathbb{P}^{k-1} \to \mathbb{R}\mathbb{P}^n$$

whose fiber over $\ell \in \mathbb{R}\mathbb{P}^n$ is $\mathrm{Hom}(\ell, \mathbb{R}^k)$. Thus $V = (\mathbb{L}^*)^k$.

$\mathrm{Th}((\mathbb{L}^*)^k)$ *is the one point compactification of $(\mathbb{L}^*)^k$ because X is compact. Thus*

$$\mathrm{Th}((\mathbb{L}^*)^k) \cong \mathbb{R}\mathbb{P}^{n+k}/\mathbb{R}\mathbb{P}^{k-1}.$$

There is notation for $\mathbb{R}\mathbb{P}^{n+k}/\mathbb{R}\mathbb{P}^{k-1}$. Define $\mathbb{R}\mathbb{P}^{n+k}_k = \mathbb{R}\mathbb{P}^{n+k}/\mathbb{R}\mathbb{P}^{k-1}$.

In total, we obtain

$$\mathbb{D}(\mathbb{R}\mathbb{P}^n_+) \cong \Sigma\mathbb{R}\mathbb{P}^{-1}_{-(n+1)}.$$

References

[At] M. Atiyah, *Thom Complexes*, Proc. London Math. Soc (3) 11, 1961, 291-310.

[H] Allen Hatcher, *Algebraic Topology.*

[M] Haynes Miller, *Vector Fields on spheres, etc. (course notes).*