Characteristic classes Problem Set 3
Due in class Wednesday October 30.

Hand in solutions to **four** of the following problems. You are encouraged to collaborate on homework assignments. Just remember to write up your proofs separately and to acknowledge your collaborators on your work. If you're not sure what a question means, please ask.

1 Reading

Milnor and Stasheff Chapters 6 and 7. (Chapter 14 may help too.)

2 Problems

1. (Milnor–Stasheff 7-C) Let B be a paracompact topology space. Let $E \to B$ be an $\mathbb{R}^m$-vector bundle and $V \to B$ be an $\mathbb{R}^n$-vector bundle. Show there is a universal formula of the form

$$w(E \otimes V) = p_{m,n}(w_1(E), \dots, w_m(E), w_1(V), \dots, w_n(V))$$

where $p_{m,n}$ is a polynomial in $m+n$ variables which is characterized as follows. If $\sigma_1, \dots, \sigma_m$ are the elementary symmetric functions of interderminants $t_1, \dots, t_m$ and $\sigma'_1, \dots, \sigma'_m$ are the elementary symmetric functions of interderminants $t'_1, \dots, t'_m$ then

$$p_{m,n}(\sigma_1, \dots, \sigma_m, \sigma'_1, \dots, \sigma'_n).$$

There is a useful hint in Milnor–Stasheff.

2.

1. Suppose that X is a CW-complex with only even dimensional cells. Show $H^*(X, \mathbb{Z})$ is free.
2. Say that X has an *affine stratification* if X can be expressed as a disjoint union of locally closed subsets U_i such that U_i is homeomorphic to $\mathbb{C}^{n_i}$ and the closure of U_i is the union of U_j's . (See Eisenbud-Harris *3264 and all that* page 27) Show that if X has a finite affine stratification, then $H^*(X, \mathbb{Z})$ is free.
3. Use Schubert cells to construct an affine stratification on the complex Grassmannians $\mathbb{C}\,\mathrm{Gr}(n, m)$.

3 (cf. Milnor–Stasheff 7-B) Use the tautological bundle and quotient bundle to give an isomorphism

$$H^*(\mathbb{C}\,\mathrm{Gr}(n, n+m), \mathbb{Z}) \cong \mathbb{Z}[c_1, \dots, c_m, \bar{c}_1, \dots, \bar{c}_m]/\langle (1 + c_1 + \ldots + c_m)(1 + \bar{c}_1 + \ldots + \bar{c}_m) - 1 \rangle$$

4 Let $S \to \mathbb{C}\,\mathrm{Gr}(2, 4)$ denote the tautological bundle on the complex Grassmannian of lines in $\mathbb{P}^3$. Find $c_4(\oplus_{i=1}^4 S^* \wedge S^*)$ (It can be shown that this is the number of lines meeting 4 lines in space.) You may assume problem 3.

5 Milnor–Stasheff 6-C.

6 Milnor–Stasheff 6-B.

7 Let $S \to \mathrm{Gr}(2, 5)$ denote the tautological bundle on the complex Grassmannian of lines in $\mathbb{P}^4$. Compute $c_6(\mathrm{Sym}^2 S^* \oplus \mathrm{Sym}^2 S^*)$. (It can be shown that this is the number of lines on a degree 4 Del Pezzo surface.)

8 Milnor–Stasheff 14-B.

9 Milnor–Stasheff 7-A.