Characteristic classes Problem Set 2
Due in class Wednesday October 16.

Hand in solutions to **four** of the following problems. You are encouraged to collaborate on homework assignments. Just remember to write up your proofs separately and to acknowledge your collaborators on your work. If you're not sure what a question means, please ask.

1 Reading

Milnor and Stasheff Chapter 5. Hatcher *Vector bundles* Section 3.1.

2 Problems

1. For a vector space V of dimension d, let $\mathrm{Gr}(n, V)$ denote the Grassmannian of n-dimensional subspaces of V containing the origin. Give a canonical isomorphism

$$\mathrm{Gr}(n, V) \cong \mathrm{Gr}(d-n, V)$$

and compute the pullback of the tautological bundle under your isomorphism.

2. (cf. Milnor and Stasheff Problem 5-E.)

1. Suppose B is compact and Hausdorff. Let $V \to B$ be a topological $\mathbb{R}^n$-bundle. Show that there exists a vector bundle $E \to B$ such that $V \oplus E$ is trivial.
2. Let $\mathcal{O}(-1) \to \mathbb{RP}^\infty$ denote the tautological bundle. Show that there does not exist a vector bundle $E \to B$ such that $\mathcal{O}(-1) \oplus E$ is trivial. (Hint: Use can use Stiefel-Whitney classes.)

3 Milnor and Stasheff Problem 5-B

4 Let L_1 and L_2 be line bundles on B. Show that $w_1(L_1 \otimes L_2) = w_1(L_1) + w_1(L_2)$ and $c_1(L_1 \otimes L_2) = c_1(L_1) + c_1(L_2)$.

5 Let $S \to \mathbb{C}\,\mathrm{Gr}_n$ denote the tautological bundle of $\mathbb{C}\,\mathrm{Gr}_n$. Compute the Chern classes of

1. $\det S$.
2. Let $n = 3$. $\mathrm{Sym}^2 S$.

Hint: You may use the result of the previous exercise and the splitting principle.

6

1. Let $U(n)$ denote the group of unitary n by n matrices. Using the fiber bundle

   $$U(n-1) \to U(n) \to S^{2n-1}$$

   given by evaluation on a unit vector, show that $H^i(U(n); \mathbb{Z}) \to H^i(U(n-1); \mathbb{Z})$ is surjective for $i \leq 2n-3$.
2. Using induction and the Leray-Hirsch theorem, show that $H^*(U(n); \mathbb{Z})$ is isomorphic to

   $$H^*(U(n); \mathbb{Z}) \cong \Lambda_{\mathbb{Z}}[x_1, x_3, \ldots, x_{2n-1}]$$

   the exterior algebra on generators x_i of odd dimension i.

In other words, up to cohomology, $U(n)$ looks like the product of spheres

$$S^1 \times S^3 \times \ldots \times S^{2n-1}$$

resulting from evaluation on basis elements.

The *classifying space* $BU(n)$ of $U(n)$ is the complex Grassmannian $\mathbb{C}\,\mathrm{Gr}_n$.

7 (see Hatcher Algebraic Topology Exercise 4.D 3) Let $V_n(\mathbb{C}^m)$ denote the space of unitary bases of n-dimensional subspaces of $\mathbb{C}^n$, giving a principal $U(n)$ bundle $V_n(\mathbb{C}^m) \to \mathrm{Gr}(n, m)$. Use the Leray-Hirsch theorem and the previous exercise to show that

$$H^*(V_n(\mathbb{C}^m); \mathbb{Z}) \cong \Lambda_{\mathbb{Z}}[x_{2m-2n+1}, x_{2m-2n+3}, \ldots, x_{2m-1}].$$

8 (Problems 8 and 9 are inspired by H. Larson and I. Vogt's paper on counting bitangents to plane quartics.) We work in complex manifolds. Let $(\mathbb{P}^2)^*$ denote the space of lines in $\mathbb{P}^2$, so $(\mathbb{P}^2)^* = \mathrm{Gr}(2, 3)$, and in fact $(\mathbb{P}^2)^*$ is isomorphic to $\mathbb{P}^2$ (as in problem 1). Let $S \to (\mathbb{P}^2)^*$ denote the tautological bundle. Let $X = \mathbb{P}(\mathrm{Sym}^2 S^*)$, so X is topologically a fiber bundle over $\mathbb{CP}^2$ with fiber $\mathbb{CP}^2$. X is the moduli space of pairs (L, Z) where L is a line in $\mathbb{P}^2$ and Z is a degree 2 divisor on L, which means a 2-point subset where the two points are allowed to be equal. Find $H^*(X, \mathbb{Z})$, the cohomology of X with coefficients in $\mathbb{Z}$.

9 Let $\mathcal{O}_X(-1)$ denote the tautological bundle of X. Let $\pi : X \to (\mathbb{P}^2)^*$ denote the projection and $\mathcal{S} = \pi^* S$. There is a short exact sequence

$$0 \to \mathcal{O}_X(-2) \hookrightarrow \mathrm{Sym}^4 \mathcal{S}^* \to V \to 0$$

such that the fiber of V over (L, Z) is the space of degree 4 polynomials on L quotiented by the span of the square of the degree 2 polynomial determining Z.

1. Using problem 8, find the Chern classes of V.
2. By Poincaré duality, c_4 can be identified with an integer. Which integer is this?

(This counts the bitangents to a degree 4 plane curve.)