Characteristic classes Problem Set 1
Due in class September 23.

Hand in solutions to **four** of the following problems. You are encouraged to collaborate on homework assignments. Just remember to write up your proofs separately and to acknowledge your collaborators on your work. If you're not sure what a question means, please ask.

1 Reading

Milnor and Stasheff Chapter 4. We skipped over Chapters 2 and 3, but we are using the material there.

2 Problems

1. Let $X = S^1 \times S^1$. Compute the Steifel-Whitney classes of TX. X embeds into $\mathbb{R}^3$ as a doughnut. Compute the Stiefel-Whitney classes of NX.

2. (Milnor and Stasheff Problem 3-B.) Given topological vector bundles $E \subset V$ define the *quotient bundle* V/E and prove that it is locally trivial. If V has a Euclidean metric, show that $E^{\perp} \cong V/E$.

3 A bilinear form on a vector space V over a field k is a bilinear map

$$\beta : V \times V \to k.$$

β is *non-degenerate* if the induced map $V \to \mathrm{Hom}(V, k)$ is an isomorphism, or equivalently, if the Gram matrix $\beta(v_i, v_j)_{i,j=1,\ldots,n}$ has non-zero determinant, where $\{v_1, \ldots, v_n\}$ is a basis of V.

1. Define the notion of a bilinear form on a vector bundle, valued in a line bundle.
2. (Milnor and Stasheff Problem 3-D) Show that if a vector bundle E possesses a Euclidean metric, then E is isomorphic to its dual bundle $\mathrm{Hom}(E, \underline{\mathbb{R}})$, where $\underline{\mathbb{R}}$ denotes the trivial bundle.

4 The degree d Veronese embedding is a map

$$\mathbb{P}^n \to \mathbb{P}^m$$

for $m = \binom{n+d}{d} - 1$ given in homogeneous coordinates by sending $[x_0 : x_1 : \ldots : x_n]$ to all possible monomials of degree d

1. Find the pullback of the tautological bundle.
2. Find the Stiefel-Whitney classes of the normal bundle.

5 Show that every short exact sequence of topological vector bundles on a CW-complex splits, i.e., given

$$0 \to V \to E \to W \to 0,$$

there is an isomorphism $E \cong V \oplus W$.

6 Milnor and Stasheff Problem 4-B

7 Milnor and Stasheff Problem 4-C

8 Milnor and Stasheff Problem 4-D

9 Milnor and Stasheff Problem 4-E