

Lecture 9

THM [Stiefel Whitney classes]

$\exists!$ w_0, w_1, \dots , assigning to each top \mathbb{R} -vector bundle $E \rightarrow B$ cohomology classes $w_i(E) \in H^i(B; \mathbb{Z}_2)$ s.t.

(1) $w_i(f^*E) = f^*(w_i(E))$ Naturality

(2) $w(E \oplus E') = w(E) \cup w(E')$
 where $w(E) = \sum_i w_i(E)$ Whitney sum formula.

(3) $w_0(E) = 1, w_i(E) = 0$ for $i > \text{rank}(E)$

(4) $w_1(S) \neq 0$ if $S \rightarrow \mathbb{R}P^1$ tautological bundle.

The proof for both theorems is the same, so we will be using Chern class notation.

$H^*(\mathbb{C}P^n; \mathbb{Z}) \cong \mathbb{Z}[x] / \langle x^{n+1} \rangle$

$H^*(\mathbb{R}P^\infty; \mathbb{Z}) \cong \mathbb{Z}[x]$
 where we choose the unique x restricting to a positive generator in $H^*(\mathbb{C}P^1; \mathbb{Z})$

THM [Chern classes]

$\exists!$ c_0, c_1, \dots , assigning to each top \mathbb{R} -vector bundle $E \rightarrow B$ cohomology classes $c_i(E) \in H^i(B; \mathbb{Z}_2)$ s.t.

(1) $c_i(f^*E) = f^*(c_i(E))$ Naturality

(2) $c(E \oplus E') = c(E) \cup c(E')$
 where $c(E) = \sum_i c_i(E)$

(3) $c_0(E) = 1, c_i(E) = 0$ for $i > \text{rank}(E)$

(4) $c_1(S) = x$ where $x \in H^2(\mathbb{C}P^1; \mathbb{Z})$
 $x \in H^2(\mathbb{C}P^1; \mathbb{Z}) \cong \mathbb{Z}[x] / \langle x^2 \rangle$

Let $E \rightarrow B$ be a rank n vector bundle

Defn $\mathbb{P}E \rightarrow B$ is defined

$\mathbb{P}E = E \setminus \text{zerosection}$
 $v \sim \lambda v$

for all $\lambda \in \mathbb{C}^*$

this is a fiber bundle w/ fiber $\mathbb{R}P^{n-1}$

Let $\mathcal{O}(-1) \rightarrow \mathbb{P}E$ be the tautological line bundle.

$\mathcal{O}(-1)_{[v]} = \mathbb{C}v$ is the fiber

By classification of topological \mathbb{C} -vector bundles

$\mathcal{O}(-1)$ gives us a map $f: \mathbb{P}E \rightarrow \mathbb{R}P^\infty$ that is unique up to homotopy s.t. $f^* S_{\mathbb{R}P^\infty} = \mathcal{O}_{\mathbb{P}E}(-1)$

$$\begin{array}{ccc}
 f^* S_{\mathbb{P}^\infty} = \mathcal{O}(-1) & \rightarrow & S_{\mathbb{P}^\infty} \\
 \downarrow & & \downarrow \\
 \mathbb{P}E & \xrightarrow{f} & \mathbb{P}^\infty \\
 \downarrow & & \\
 E & \rightarrow & B
 \end{array}$$

Let $t \in H^2(\mathbb{P}E; \mathbb{Z})$ be the pullback $f^*x = t$

Note f is associated to a map $\tilde{f}: \mathcal{O}(-1) \rightarrow \mathbb{C} \oplus \infty$ which is linear and injective on each fiber & $\tilde{f}|_{E_b}$ is linear & injective for all $b \in B$.

$$f^* \Big|_{\mathbb{P}E_b} (x) = t \Big|_{\mathbb{P}E_b} = t_b$$

is a generator for

$$H^*(\mathbb{P}E_b; \mathbb{Z}) \cong \mathbb{Z}[t_b] / \langle t_b^n \rangle$$

We can now apply Leray Hirsch!

$\{1, t, t^2, t^3, \dots, t^{n-1}\} \in H^*(\mathbb{P}E)$ restricts to a \mathbb{Z} -basis $H^*(\mathbb{P}E)$ for all $b \in B$.

Leray-Hirsch

$$\Rightarrow H^*(E) = \bigoplus_{i=0}^{n-1} H^*(B) t^i$$

free module deg $2i$

Then \exists unique $c_i \in H^{2i}(B)$ s.t.

$$t^n + c_1(E) t^{n-1} + c_2(E) t^{n-2} + \dots + c_n(E) = 0$$

Define $c_0(E) = 1$ and $c_i(E) = 0$ for $i > n$.

CLAIM: This assignment $c_i(E)$ satisfies (1) & (4) in our thm on the construction of characteristic classes.

(1) Naturality:

Given $g: B' \rightarrow B$

$$\begin{array}{ccccc}
 \tilde{g}^* \mathcal{O}(-1) & \xrightarrow{\tilde{g}} & \mathcal{O}(-1) & & \\
 \downarrow & & \downarrow & & \\
 g^* \mathbb{P}E & \rightarrow & \mathbb{P}E & \xrightarrow{f} & \mathbb{P}^\infty \\
 \downarrow & & \downarrow & & \\
 B' & \xrightarrow{g} & B & &
 \end{array}$$

$$g \circ f: g^* \mathbb{P}E \rightarrow \mathbb{P}^\infty$$

$$(g \circ f)^* x = g^* t = x$$

$$(\tilde{q}f)^* S_{\mathbb{P}^\infty} \simeq q^* \mathcal{O}(-1)_{q^* \mathbb{P}^E}$$

$$\Rightarrow \tilde{q}^* c_i(E) = c_i(q^* E)$$

(3) Given b/c we defined $G(E) = 1$ & $c_i(E) = 0$ $i > n$.

(2) Whitney sum
Consider $E_1 \oplus E_2$.

Since $E_i \hookrightarrow E_1 \oplus E_2$
 $\mathbb{P}E_i \longrightarrow \mathbb{P}(E_1 \oplus E_2)$

Define

$$U_i = \mathbb{P}(E_1 \oplus E_2) \setminus \mathbb{P}E_i$$

$$= \{ [x_1, \dots, x_m, y_1, \dots, y_n] \mid$$

$$(x_1, \dots, x_m) \in E_1,$$

$$(y_1, \dots, y_n) \in E_2 \text{ and}$$

$$\text{not all } y_i \text{ are zero} \}$$

Note that

$\mathbb{P}E_2 \subset U_1$ and in fact
 $\mathbb{P}E_2$ is homotopic equivalent
to U_1

$$U_1 \longrightarrow \mathbb{P}E_2$$

$$[x_1, \dots, x_m, y_1, \dots, y_n] \mapsto [0, \dots, 0, y_1, \dots, y_n]$$

is a deformation retraction

$$\mathbb{P}E_2 \hookrightarrow \mathbb{P}(E_1 \oplus E_2)$$

$$U_1 \longrightarrow \mathbb{P}E_2 \longrightarrow U_1$$

$$[x_1, \dots, x_m, y_1, \dots, y_n] \mapsto [0, \dots, 0, y_1, \dots, y_n]$$

which is homotopic to the identity

w/ the homotopy

$$[x_1, \dots, x_m, y_1, \dots, y_n]$$

$$\downarrow$$

$$[tx_1, \dots, tx_m, y_1, \dots, y_n]$$

Similarly

$\mathbb{P}E_1$ is homotopy equiv to U_2

$$U_2 := \{ [x_1, \dots, x_m, y_1, \dots, y_n] \mid \text{not all } x_i \text{ are zero} \}$$

$$\Rightarrow \mathbb{P}(E_1) \cap \mathbb{P}(E_2) = \emptyset \text{ and,}$$

$$\mathbb{P}(E_1) \cup \mathbb{P}(E_2) \simeq U_1 \cup U_2 \simeq \mathbb{P}(E_1 \oplus E_2)$$

Choose $f: \mathbb{P}(E_1 \oplus E_2) \rightarrow \mathbb{P}^\infty$ classifying $\mathcal{O}_{\mathbb{P}(E_1 \oplus E_2)}(-1)$

$\Rightarrow f|_{\mathbb{P}(E_i)}$ classifies $\mathcal{O}_{\mathbb{P}(E_i)}(-1)$

Let $t = f^* x \in H^*(\mathbb{P}(E_1 \oplus E_2))$

$$b_1 = t^m + c_1(E_1)t^{m-1} + \dots + c_m(E_1)$$

$$b_2 = t^n + c_1(E_2)t^{n-1} + \dots + c_n(E_2)$$

$$\Rightarrow b_1 \cdot b_2 = 0$$

$b_1|_{\mathbb{P}(E_1)} = 0$ $b_2|_{\mathbb{P}(E_2)} = 0$
 by defn of chern classes

$\Rightarrow b_1$ is the image of
 a class in
 $H^*(\mathbb{P}(E_1 \oplus E_2), \mathbb{P}(E_1))$
 $\simeq H^*(\mathbb{P}(E_1 \oplus E_2), U_2)$

and b_2 is the image of
 a class in
 $H^*(\mathbb{P}(E_1 \oplus E_2), \mathbb{P}(E_2))$
 $\simeq H^*(\mathbb{P}(E_1 \oplus E_2), U_1)$

$\Rightarrow b_1 \cup b_2$ is the image
 of a class in

$$H^*(\mathbb{P}(E_1 \oplus E_2), U_1 \cup U_2) \\ \simeq H^*(\mathbb{P}(E_1 \oplus E_2), \mathbb{P}(E_1 \oplus E_2)) \\ \simeq 0$$

$$\Rightarrow b_1 \cup b_2 = 0$$

$$\Rightarrow t^{m+n} + (c_1(E_1) + c_1(E_2))t^{m+n-1} + \dots \\ + \dots = 0$$

and by uniqueness we
 get $c_i(E_1 \oplus E_2) = \sum_{j=1}^{\min} c_j(E_1) c_{i-j}(E_2)$

(4) we want to
 show that $c_1(S) = -t$

Note that for $S \rightarrow \mathbb{C}P^1$

$$\mathbb{P}S \simeq \mathbb{C}P^1$$

$$S_{\mathbb{P}^1} = \mathcal{O}_{\mathbb{P}^1}(-1) \rightarrow S_{\mathbb{P}^1 \times \infty}$$

$$\downarrow \qquad \qquad \qquad \downarrow \\ \mathbb{C}P^1 = \mathbb{P}S \longrightarrow \mathbb{P}^\infty$$

$$\downarrow \qquad \qquad \qquad \parallel \\ \mathbb{C}P^1 \longrightarrow \mathbb{P}^\infty$$

$$\Rightarrow t + c_1(S) = 0 \Rightarrow c_1(S) = -t$$