

## Lecture 8

Notation Let  $\mathcal{O}$  denote the trivial bundle of rank 1.

Recall:

$$0 \rightarrow S \rightarrow \mathcal{O}^m \rightarrow Q \rightarrow 0$$

$$\& \text{Hom}(S, Q) \cong T\text{Gr}(n, m)$$

Let  $V \rightarrow B$  be a rank  $R$ -vector bundle. If we have a map

$$f: V \rightarrow V$$

→ we have an induced map

$$\Lambda^r V \xrightarrow{\Lambda^r f} \Lambda^r V$$

$$\Lambda^r f \in \text{Hom}(\Lambda^r V, \Lambda^r V) \cong \Lambda^r V^* \otimes \Lambda^r V^* \\ \cong \mathcal{O}$$

$\Lambda^r f$  is a global section of  $\mathcal{O}$   
seeing  $\text{Hom}(\Lambda^r V, \Lambda^r V)$  as a vector bundle

i.e.  $\Lambda^r f$  is a function

$$\Lambda^r f(b) = \det(V_b \xrightarrow{f_b} V_b)$$

$$\text{Let } \det V := \Lambda^r V$$

Fact for any smooth manifold  $X$   
(or scheme)

$\det T^* X$  is an orientation  
bundle (or sheaf) and is connected to  
Poincaré & Serre duality.

Plücker embedding

$$\text{Gr}(n, m) \hookrightarrow \mathbb{P}(\Lambda^n \mathcal{O}_{\text{pt}}^m)$$

$\text{span}\{v_1, \dots, v_n\} \rightarrow v_1 \wedge \dots \wedge v_n$   
n linearly  
independent  
vectors

$$\mathbb{P}(\Lambda^n \mathcal{O}_{\text{pt}}^m) \cong \mathbb{P}^{\binom{m}{n}-1}$$

$$\text{and } S \rightarrow \mathbb{P}^{\binom{m}{n}-1}$$

Then define

$$\mathcal{O}_{\text{Gr}}(1) \rightarrow \text{Gr}(n, m)$$

$$\mathcal{O}_{\text{Gr}}(1) = p^* S_{\mathbb{P}^{\binom{m}{n}-1}}^*$$

Q How can we express  
 $\mathcal{O}_{\text{Gr}}(1)$  in terms of  
 $S_{\text{Gr}}$ ?

$$\mathcal{O}(1) = p^* S_{\mathbb{P}^{\binom{m}{n}-1}}^*$$

$$= \Lambda^n S_{\text{Gr}}^*$$

Note:

$$S_{\text{Gr}}^*$$

$$\downarrow \pi$$

$$\text{Gr}(n, m) \xrightarrow{P} \mathbb{P}(\Lambda^n \mathcal{O}_{\text{pt}}^m) \cong \mathbb{P}^{\binom{m}{n}-1}$$

$$S_{\mathbb{P}^{\binom{m}{n}-1}}$$

$$\downarrow$$

$$\mathbb{P}^{\binom{m}{n}-1}$$

$$\pi^{-1}([v_1, \dots, v_n]) = \text{span}\{v_1, \dots, v_n\}$$

$\downarrow p$

$$v_1 \wedge \dots \wedge v_n$$

& the fiber of  $p^* S_{\mathbb{P}^{(m)-1}}^*$   
at  $[v_1, \dots, v_n]$  is  $v_1 \wedge \dots \wedge v_n$

$\Rightarrow$

2) what is  $\det(S_{\mathbb{P}^{(m)-1}}^*)$ ?

Note

$$T\text{Gr} \simeq \text{Hom}(S_{\text{Gr}}, Q)$$

$$= S_{\text{Gr}}^* \otimes Q$$

$$\det T\text{Gr} = (\det S^*)^{m-n} \otimes (\det Q)^n$$

$$0 \rightarrow S \rightarrow \Theta^m \rightarrow Q \rightarrow 0$$

$$\det S \otimes \det Q \simeq \det \Theta^m \\ \simeq \underline{\mathbb{R}^m}$$

$$\Rightarrow \det Q \simeq \det S^*$$

$$\Rightarrow \det(T\text{Gr}) \simeq (\det S^*)^{m-n} \otimes (\det S^*)^n \\ \simeq (\det S^*)^m$$

$$\Rightarrow \det T^*\text{Gr} \simeq (\det S)^m \simeq \Theta(-1)^m =: \Theta(-m)$$

notation  
this is the canonical bundle  
in algebraic geometry

### CONSTRUCTION OF STIEFEL WHITNEY & CHERN CLASSES for topological vector bundles ( $\mathbb{R}, \mathbb{C}$ )

Defn A fiber bundle with fiber  $F$  is a map  
 $p: E \rightarrow B$  s.t.  $B$  admits an open cover  $\{U_\alpha\}_{\alpha \in A}$  and trivializations  $p^{-1}(U_\alpha) \simeq U_\alpha \times F$

$$p^{-1}(U_\alpha) \simeq U_\alpha \times F$$

$\downarrow \text{proj}$

$U_\alpha$

We want to compute  $H^*(E)$   
Let  $R$  be a ring, cohomology  
of a product:  
 $H_n(C_1) \otimes H_m(C_2) \rightarrow H_{n+m}(C_1 \otimes C_2)$   
this is not an isomorphism,  
Tor gives the error term.

Then,

$$0 \rightarrow \bigoplus_{i=0}^n (H^i(B; R) \otimes H^{n-i}(F; R)) \rightarrow H^*(B \times F; R) \rightarrow \bigoplus_{i=0}^{n-1} \text{Tor}(H^i(B; R) \otimes H^{n-i-1}(F; R)) \rightarrow 0$$

If  $H^*$  is free (e.g. if  $R$  is a field)

$$\Rightarrow H^*(B; R) \otimes H^*(F; R) \xrightarrow{\sim} H^*(B \times F; R)$$

THM [Leray-Hirsch] Let  $B$  be a CW complex, and  $p: E \rightarrow B$  be a fiber bundle with fiber  $F$ .

Suppose (1)  $H^i(F; R) \cong R$  mi free and of finite rank

(2)  $\exists \{c_j\}_{j \in J}$   
 $c_j \in H^{n-j}(E; R)$  which restrict to an  $R$ -module basis

of  $H^*(p^{-1}(b); R) \quad \forall b \in B$   
 $\Rightarrow H^*(E; R)$  is the free  $H^*(B; R)$  module on the basis  $\{c_j\}_{j \in J}$

RMK  
\*  $H^*(F; R)$  is free and of finite rank, the hypothesis

holds for  $B \times F$ .

\* It does not hold for the Hopf fibration with fiber  $S^1$

$$h: S^3 \rightarrow S^2 \cong \mathbb{P}^1$$

$$\{(x, y) \in \mathbb{C}^2 \mid |x|^2 + |y|^2 = 1\} \hookrightarrow [x:y]$$

Proof:

Case 1:  $B$  has finitely many cells ( $B$  is a finite CW cx)

By induction on the number of cells

Let  $B$  be formed by attaching a 1-cell to  $B'$

$$\Rightarrow \text{by induction} \quad H^*(B) \otimes H^*(F) \xrightarrow{\sim} H^*(p^{-1}(B'))$$

$$E' = p^{-1}(B')$$

$$r \otimes \sum r_j c_j \mapsto \sum r_j c_j$$

Consider the LES in relative cohomology

$$\dots \rightarrow H^*(B, B') \otimes H^*(F) \rightarrow H^*(B) \otimes H^*(F) \rightarrow H^*(B') \otimes H^*(F) \rightarrow \dots$$

$$\downarrow \textcircled{2} \simeq$$

$$\downarrow \textcircled{3}$$

$\downarrow \textcircled{1}$  by induction

$$\dots \rightarrow H^*(E, E') \longrightarrow H^*(E) \longrightarrow H^*(E) \rightarrow \dots$$

$$\textcircled{2} \quad E/E' \simeq \sum \times F$$

THM 4D1 in Hatcher

$$p^{-1}(e_n) \simeq e_n \times F$$

$$\downarrow e_n$$

$$E/E' \underset{\text{excision}}{\simeq} p^{-1}(e_n) / p^{-1}(\partial e_n)$$

$$\Rightarrow \sum F_+ = S^n \wedge F_+ \simeq \mathbb{Q}^n \times F / \partial \mathbb{Q}^n \times F$$

Case 2: (Cohomology turns colimits into inverse limits)

$B^{(i)}$  denotes the  $i$ th skeleton

$$B = \varinjlim_i B^{(i)}$$

$$H^*(B; \mathbb{Q}) = \varprojlim_i H^*(B^{(i)}; \mathbb{Q})$$

Similarly for  $p^{-1}(B^{(i)})$  and we are done.  $\square$