

## Lecture 7

**Proposition** For  $B$  paracompact

Let  $\pi: E \rightarrow B$

be an  $\mathbb{R}^n$  (or  $\mathbb{C}^n$  resp)  
topological vector bundle

$\Rightarrow$  There exists

$f: B \rightarrow \text{Gr}_n$

with  $f^*S \cong E$

Let  $V \xrightarrow{\pi} B \times [0,1]$  be  
a rank  $n$  topological vector  
bundle

**LEMMA** If

$V|_{\pi^{-1}(B \times [0,a])}$  and  
 $V|_{\pi^{-1}(B \times [a,1])}$  are  
trivial  $\Rightarrow V$  is trivial

Pf let

$$h_0: V|_{\pi^{-1}(B \times [0,a])} \xrightarrow{\sim} B \times [0,a] \times \mathbb{R}^n$$

$$h_1: V|_{\pi^{-1}(B \times [a,1])} \xrightarrow{\sim} B \times [a,1] \times \mathbb{R}^n$$

be trivializations

$\Rightarrow$  define  $A: B \times \mathbb{R}^n \rightarrow \mathbb{R}^n$  by  
 $B \times \mathbb{R}^n \xrightarrow{\quad \text{project by} \quad} B \times \{a\} \times \mathbb{R}^n \xrightarrow{\quad \text{proj} \quad} \mathbb{R}^n$   
 $(b,v) \longmapsto h_0 h_1^{-1}(b, a, v)$

By abuse of notation write that

$$h_1(v) = (b, t, \underset{n}{\underbrace{h_1(v)}})$$

Suppose that

$$v \in V|_{b \times t}$$

for  $t \in [a,1]$

Then define

$$\tilde{h}_1: V|_{\pi^{-1}(B \times [a,1])} \xrightarrow{\quad ? \quad}$$

$$B \times [a,1] \times \mathbb{R}^n$$

$$\tilde{h}_1(v) = (b, t, A(b, t, v))$$

**RMK:** The pf above does  
not work smoothly or  
algebraically. Furthermore  
the lemma is not even true  
in the algebraic category  
for an arbitrary case

Now, **PROPOSITION**  
 $B$  compact,

$V \xrightarrow{\pi} B \times [0,1]$  rank  $n$  top  
vector bundle

Then

$$V|_{\pi^{-1}(B \times \{0\})} \cong V|_{\pi^{-1}(B \times \{1\})}$$

$[0,1]$  compact  
 $\Rightarrow$  for each  $b \in B$  we can find  $0 = t_0 < t_1 < \dots < t_r = 1$  and neighborhoods  $U_{b,i}$  s.t.  $V|_{U_{b,i} \times [t_{i-1}, t_i]}$  is trivial.

$$U_b = \bigcap_i U_{b,i}$$

is an open neighborhood of  $b$  in  $B$

By the lemma above

$V|_{U_b \times [0,1]}$  is trivial

Since  $B$  compact we can choose finitely many  $\{U_\alpha\}$  covering  $B$  s.t.

$V|_{U_\alpha \times [0,1]}$  is trivial

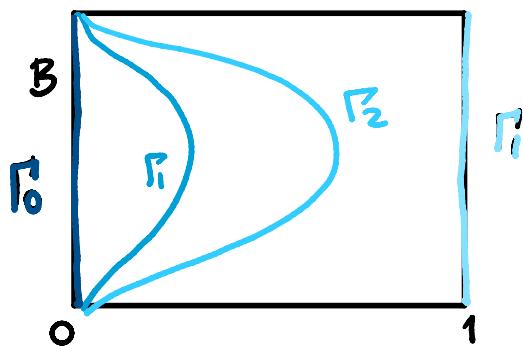
Choose a partition of unity  $\{\varphi_i : B \rightarrow [0,1]\}$  subordinate to  $U_\alpha$

$$\text{let } \Psi_i = \varphi_1 + \dots + \varphi_i$$

$$\Psi_0 = 0$$

$$\Psi_r = 1$$

$$\Gamma_i = \text{graph of } \Psi_i = \{(b, \Psi_i(b))\}$$



$$\text{let } V_i = V|_{U_i \times [0,1]}$$

$$V_i \rightarrow \begin{matrix} \Gamma_i \\ \downarrow \\ B \end{matrix}$$

$$\Gamma_i \rightarrow \Gamma_{i+1}$$

$$(b, \Psi_i(b)) \rightarrow (b, \Psi_{i+1}(b))$$

lifts to an isomorphism

The composition

$$V_0 \rightarrow V_1 \rightarrow \dots \rightarrow V_n$$

$$\Downarrow$$

$$V|_{B \times [0,1]}$$

gives the desired isomorphism

( $B$  paracompact pf is analogous)

Defn Two maps

$f, g: X \rightarrow Y$  are homotopic if  $\exists$  a homotopy  $H: X \times [0, 1] \rightarrow Y$

s.t.

$$H|_{X \times \{0\}} = f$$

$$H|_{X \times \{1\}} = g$$

Proposition Let  $E \rightarrow Y$  be a topological vector bundle. If  $f \sim g \Rightarrow f^*E \cong g^*E$

Pf

$$f^*E \cong H^*E|_{X \times \{0\}}$$

$$\cong H^*E|_{X \times \{1\}}$$

$$\cong g^*E$$

Notation: Let

$[X, Y]$  = homotopy classes of maps  $X \rightarrow Y$ .

Let  $\text{Vect}^n(X) = \{$  isomorphism classes of rank  $n$  vector bundles on  $X\}$

We have a well defined map

$$[X, \text{Gr}_n] \rightarrow \text{Vect}^n(X)$$

for  $X$  paracompact

THM classification of two vector bundles  $X$  paracompact

The map

$$[X, \text{Gr}_n] \rightarrow \text{Vect}^n(X)$$

$$f \mapsto f^*S$$

is a bijection.

Pf During the last

lecture we showed surjectivity. Showed its well defined. NTS injectivity

$$\text{If } f_1^*S \cong f_2^*S$$

$$\Rightarrow f_1 \sim_{\text{homtpy}} f_2$$

Suppose  $f, g: X \rightarrow \text{Grn}$  &  
 $f^*S \cong g^*S \cong E$   
By defn of Grn  
 $\Rightarrow \tilde{f}, \tilde{g}: E \rightarrow \mathbb{R}^{\oplus \infty}$   
which are linear & injective  
on each fiber.

To construct a homotopy  
btwn  $f, g$  it suffices to

construct a homotopy  
btwn  $\tilde{f}$  and  $\tilde{g}$   
s.t.  $\forall t \tilde{f}_t$  is linear  
and injective on  
fibers of  $E$ .

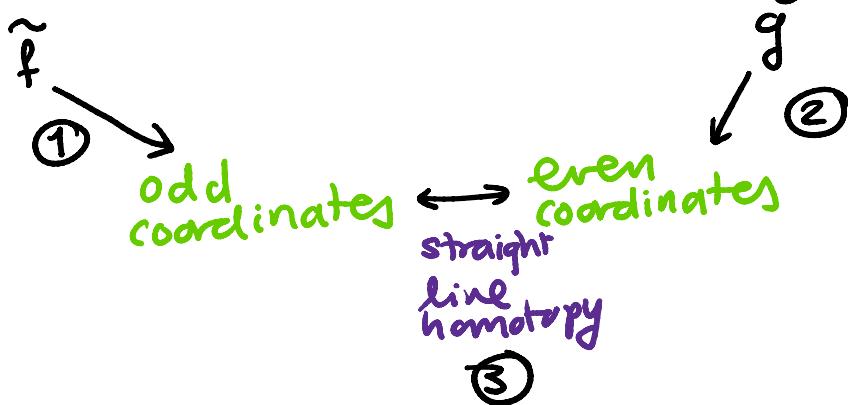
we can map a pt on  $\mathbb{R}^{\oplus \infty}$

$$(x_1, x_2, \dots) \mapsto t(x_1, x_2, \dots) + (1-t)(x_1, 0, x_2, 0, \dots)$$

This defines a homotopy btwn Id map and odd  
number coordinates only

(can also use map  
 $(x_1, \dots) \mapsto t(x_1, x_2, \dots) + (1-t)(0, x_1, 0, x_2, \dots)$ )

Then



□

COROLLARIES: Topological vector bundles over  
paracompact spaces have nondegenerate  
inner products by pulling back the Euclidean  
metric on  $\mathbb{R}^{\oplus \infty}$

Ex If  $M \hookrightarrow \mathbb{R}^n$   
embedded

$$M \rightarrow TM$$

$$m \mapsto T_m M \subset \mathbb{R}^n$$

Gauss map  
and  $f^* S \cong TM$

Consider  $H^*(\mathbb{R}\text{Gr}_n; \mathbb{Z}_2)$   
or  $H^*(\mathbb{C}\text{Gr}_n; \mathbb{Z})$

Let  $A = \left\{ \begin{array}{l} \mathbb{Z}_2, \quad \mathbb{R} \\ \mathbb{Z}, \quad \mathbb{C} \end{array} \right.$

$\Rightarrow A[t_1, \dots, t_n]$  polynomial ring with  $n$  variables

$$A[t_1, \dots, t_n]^{S^n} \subset A[t_1, \dots, t_n]$$

Symmetric polynomials

Q How can we find a map  
 $H^*(\text{Gr}_n) \rightarrow A[t_1, \dots, t_n]^{S^n}$   
using a vector bundle on  
 $\underbrace{\mathbb{P}^\infty \times \dots \times \mathbb{P}^\infty}_{n \text{ times}}$  and what  
is the image of  $w_i(s)$ ?

$(\mathbb{P}^\infty)^n \rightarrow \text{Gr}_n$   
 $(l_1, \dots, l_n) \mapsto \bigoplus_{i=1}^n l_i$   
where  $l_i$  is a line in  $\mathbb{P}^\infty$

Then  $\pi_i: (\mathbb{P}^\infty)^n \rightarrow \mathbb{P}^\infty$   
ith projection

Then, if  $S_1 \rightarrow \mathbb{P}^\infty$   
is the tautological bundle

$\pi_i^* S_1 \rightarrow (\mathbb{P}^\infty)^n$  is  
a line bundle.

and

$$V = \bigoplus_{i=1}^n \pi_i^* S_1 \rightarrow (\mathbb{P}^\infty)^n$$

is a vector bundle

By the classification  
thm of top vector  
bundles,  $\exists$   
 $f \in [(\mathbb{P}^\infty)^n, \text{Gr}_n]$   
s.t.  $f^* S \cong V$

which gives an induced  
map

$$H^*((\mathbb{P}^\infty)^n) \xrightarrow{f^*} H^*(\text{Gr}_n)$$

Note

$$H^*(\mathbb{C}P^\infty)^n \cong \bigotimes_{i=1}^n H^*(\mathbb{P}^\infty) \cong \bigotimes_{i=1}^n A[t_i]$$
$$\cong A[t_1, \dots, t_n]$$

Furthermore,  
we have an action of  $\sigma \in S_n$  on  $(\mathbb{P}^\infty)^n$  and  
 $\sigma * V \cong V$

Then  $f \circ \sigma \cong_{\text{homotopic}} f$

$\Rightarrow$  Image of  $H^*(Gr_n)$  is in symmetric polynomials  
 $w(S) \mapsto w(V)$  by naturality

$$w\left(\bigoplus_{j=1}^n \pi_j^* S\right) = \prod_{j=1}^n w(\pi_j(S))$$
$$= \prod_{j=1}^n (1 + t_j)$$