

# Lecture 5

**Defn**  $f: M \rightarrow N$ , a diff map between smooth manifolds is an **immersion** if

$$T_x M \rightarrow T_{f(x)} N$$

is injective.

An **embedding** is an injective immersion which induces a diffeomorphism onto its image

Given an immersion  $F: M \rightarrow \mathbb{R}^{n+k}$

where  $M$  is an  $n$ -mfld then we have the short exact sequence

$$0 \rightarrow TM \rightarrow F^* T\mathbb{R}^{n+k} \rightarrow N_M \mathbb{R}^{n+k} \rightarrow 0$$

Note  $F^* T\mathbb{R}^{n+k} = \mathbb{R}^{n+k}$   
this s.e.s splits so

$$\mathbb{R}^{n+k} = TM \oplus N_M \mathbb{R}^{n+k}$$

$$\omega(N_M \mathbb{R}^{n+k}) = \omega(TM)^{-1}$$

Since the rank of  $\omega(N_M \mathbb{R}^{n+k})$  is  $k$   
 $\Rightarrow \omega(TM)^{-1}$  has to vanish in degrees  $> k$ .

Thus we can find obstructions to immersions using SW classes

**Fun fact:**

Boys surface gives an immersion

$$\mathbb{R}P^2 \rightarrow \mathbb{R}^3$$

But

$\mathbb{R}P^2 \not\hookrightarrow \mathbb{R}^3$  does not embed!

**THM** If  $\mathbb{R}P^{2r}$  immerses in  $\mathbb{R}^{2r+k} \Rightarrow k \geq 2r-1$  and it is a sharp bound.

**PF**  $\omega(T\mathbb{R}P^{2r}) = (1+t)^{2r-1}$  in  $H^r(\mathbb{R}P^{2r}; \mathbb{Z}_2)$

$$\omega(T\mathbb{R}P^{2r})^{-1} = ((1+t)^{2r-1})^{-1}$$

**Aside:** The Frobenius ring endomorphism on  $\mathbb{Z}_p[t]$

maps  $a \rightarrow a^p$  and  $(a+b)^p \rightarrow a^p + b^p$

and  $(a+b)^{p^r} \rightarrow a^{p^r} + b^{p^r}$

$$\begin{aligned}
 ((1+t)^{2^r+1})^{-1} &= ((1+t)^{2^r} (1+t))^{-1} \\
 &= ((1+t^{2^r})(1+t))^{-1} \\
 &= (1+t+t^{2^r})^{-1} \text{ in } H^1(\mathbb{P}^{2^r}; \mathbb{Z}_2) \\
 &= 1 - (t+t^{2^r}) + (t+t^{2^r})^2 - \dots
 \end{aligned}$$

[WHITNEY EMBEDDING THM]  
 If  $M$  is a smooth compact  $n$ -mfd  $\Rightarrow \exists$  an embedding  $M \hookrightarrow \mathbb{R}^{2n}$

[Whitney]

If  $M$  is a smooth compact  $n$ -mfd  $\Rightarrow \exists$  an immersion  $M \rightarrow \mathbb{R}^{2n-1}$

Q Let  $Q \rightarrow \mathbb{R}P^n$  denote the quotient bundle

What is  $w(Q)$ ?

Recall  $Q := \underline{\mathbb{R}^{n+1}} / S$

$$\$ \quad 0 \rightarrow S \rightarrow \underline{\mathbb{R}^{n+1}} \rightarrow Q \rightarrow 0$$

In Top this splits

$$\Rightarrow w(\underline{\mathbb{R}^{n+1}}) = w(S)w(Q)$$

$$\begin{aligned}
 \Rightarrow w(Q) &= w(S)^{-1} w(\underline{\mathbb{R}^{n+1}}) \\
 &= (1+t)^{-1} \cdot 1
 \end{aligned}$$

## GRASSMANIANS

$\mathbb{R}Gr(n, m) := \{ \mathbb{R}^n \subset \mathbb{R}^m \text{ passing through the origin} \}$

$Gr(n, m) = \{ \text{linear } n\text{-planes in } m\text{-space} \}$   
 containing 0

defined for  $\mathbb{C}$  analytic & algebraic spaces,  $\mathbb{C}^n$  or  $\mathbb{A}_s^n$  for  $S = \text{Spectrum}(K)$   $K$  is any field.

Proposition:  $Gr(n, m)$  is a smooth manifold of dimension  $n(m-n)$

We will use  $\mathbb{R}^n$  notation for  $n$ -space but can substitute for any of the others.

Pf: Let  $X_0$  be an  $n$ -plane in  $\mathbb{R}^m$   
Choose a projection  
 $p: \mathbb{R}^m \rightarrow X_0$

An open set  $U \subset Gr(n, m)$  is given by

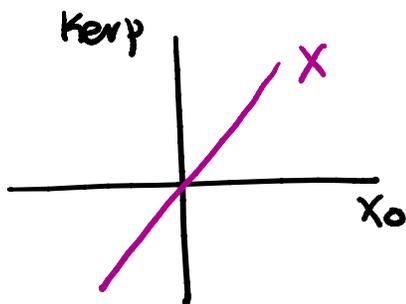
$$U = \{ X \text{ } n\text{-plane} \mid X \cap \ker(p) = \{0\} \}$$

$$\Rightarrow p|_X: X \xrightarrow{\sim} X_0 \text{ and}$$

$$U \xrightarrow[\text{isomorphism of sets}]{\sim} \text{Hom}(X_0, \ker(p))$$

$$X \mapsto \varphi_X, \text{ for } v \in X_0$$

$$\varphi_X(v) = \underbrace{(p|_X)^{-1}(v) - v}_{\in \ker(p)}$$



$X$  is a graph of a homeomorphism  $\varphi_X$

One can find w/ linear algebra arguments that

$$\text{Hom}(X_0, \ker(p)) \cong \mathbb{R}^{n(m-n)}$$

& can find transition maps & show that they are continuous (or holomorphic or algebraic or...)

Note  $T_x U \cong T_x \text{Hom}(X_0, \ker(p))$

Furthermore  $S \rightarrow Gr(n, m)$  where  $S_{X_0} = X_0$  is the tautological bundle  
and  $Q \rightarrow Gr(n, m)$  where  $Q_{X_0} = \mathbb{R}^m / X_0$

**COROLLARY**  $TGr(n, m) \cong \text{Hom}(S, Q)$

we already saw this for  $\mathbb{P}^n = Gr(1, n+1)$

$Gr(2,4) = \{ \text{space of } \mathbb{R}^2\text{'s in } \mathbb{R}^4 \}$

Let  $V$  be a vector space and its projectivization is

$$IPV = V - \{0\} / v \sim \lambda v \text{ for } v \in V \text{ and } \forall \text{ nonzero scalars } \lambda.$$

$Gr(2,4) = \text{space of } IP(\mathbb{R}^2) = IP^1 \text{ in } IP(\mathbb{R}^4) = IP^3$   
 = space of lines in 3-space including lines in the copy of  $IP^3$  at  $\infty$ .

$$IP^3 = \mathbb{R}^3 \cup IP^2$$

$\{x_0 \neq 0\}$  at  $\infty$   
 $\{x_0 = 0\}$

$$IP^3 = \{ [x_0 : x_1 : x_2 : x_3] \mid x_0, \dots, x_3 \in \mathbb{R} \}$$

What are the coordinates?

Let  $\{e_1, e_2, e_3, e_4\}$  denote the std basis of  $\mathbb{R}^4$ .

$$\text{Let } X_0 = \text{span} \{e_3, e_4\}$$

$$p: \mathbb{R}^4 \rightarrow X_0$$

$$p(x_1, x_2, x_3, x_4) = (x_3, x_4)$$

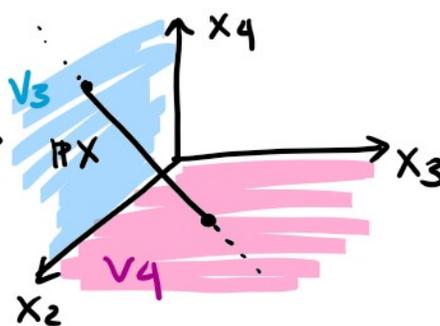
$$(a, b, c, d) \in U = \mathbb{R}^4$$

$$\updownarrow$$

$$\text{span} \{ a e_1 + b e_2 + e_3, c e_1 + d e_2 + e_4 \}$$

Note that  $IPX$  is a line in  $IP^3$

Consider  $\mathbb{R}^4$  where  $x_1 = 1$



$$x_2 x_3 \text{ plane is } V_4 = \{x_4 = 0\}$$

$$x_2, x_4 \text{ plane is } V_3 = \{x_3 = 0\}$$

Where does the line  $IPX$  corresponding to  $X$  intersect  $V_4$  &  $V_3$ ?

$$X \cap V_4 = \text{span} \{ c e_1 + d e_2 + e_3 \}$$

$$= [c : d : 0 : 1]$$

$$= \left( \frac{d}{c}, 0, \frac{1}{c} \right)$$

$$X \cap V_3 = \text{span} \{ a e_1 + b e_2 + e_3 \}$$

$$= [a : b : 1 : 0]$$

$$= \left( \frac{b}{a}, \frac{1}{a}, 0 \right)$$

Geometric perspective:

$Gr(n, m) = \text{space of } \mathbb{P}^{n-1}\text{'s}$   
in  $\mathbb{P}^{m-1}$

to put coords choose  $n$  copies  
of  $\mathbb{P}^{m-n}$ , call them  $v_1, \dots, v_n$ .  
Give  $v_i$  coordinates and  
write down the intersection  
points  $X \cap v_i$ :

$\mathbb{R}Gr(n, m)$  is compact  
 $\mathbb{C}Gr(n, m)$  is compact  
 $Gr(n, m)$  is projective  
(embeds algebraically  
in some  $\mathbb{P}^N$ )

Pf: for projective

Choose  $X \in Gr(n, m)$  &  
basis  $\{b_1, \dots, b_n\}$  for  $X$

$$b_1 \wedge \dots \wedge b_n \in \underbrace{\mathbb{R}^m \wedge \dots \wedge \mathbb{R}^m}_{n \text{ times}} \\ = \mathbb{R}^{\binom{m}{n}}$$

If we chose a different basis  
it would be

$$\{A b_1, \dots, A b_n\}$$

for  $A \in GL_n$  fixed.

Note

$$A b_1 \wedge \dots \wedge A b_n = (\det A) b_1 \wedge \dots \wedge b_n$$

$b_1 \wedge \dots \wedge b_n$  is well defined  
in  $\mathbb{P}(\underbrace{\mathbb{R}^m \wedge \dots \wedge \mathbb{R}^m}_{n \text{ times}})$

+ plücker embedding