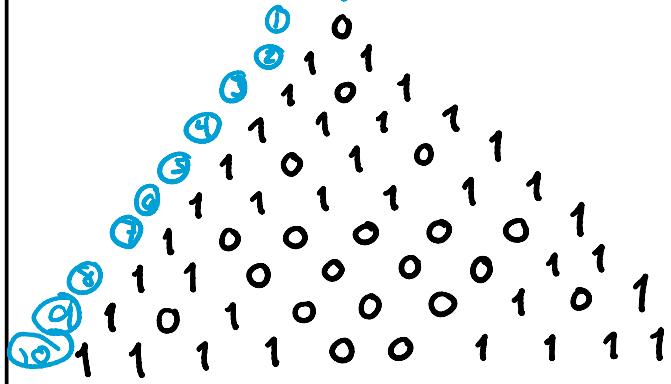


Lecture 4

(1) Find the binomial coefficients
 $\binom{n}{i} \bmod 2 \quad 0 \leq i \leq 10.$

Pascals triangle



(2) Find the inverse of
 $(1+t)^{n+1}$ in $H^*(\mathbb{RP}^n; \mathbb{Z}_2)$

Recall

$$H^*(\mathbb{RP}^n; \mathbb{Z}_2) = \{a_0 + a_1 t + \dots\}$$

where $a_i t^i \in H^i(\mathbb{RP}^n; \mathbb{Z}_2)$

$$\text{Ex } n=1 \\ (1+t)^2 = 1 \Rightarrow ((1+t)^2)^{-1} = 1$$

$$\text{Note that in } H^*(\mathbb{RP}^n; \mathbb{Z}_2) \\ t^n = 0.$$

$$n=2$$

$$(1+t)^3 = 1 + t + t^2$$

$$((1+t)^3)^{-1} = 1 + (-t + t^2) + t^2 \\ = 1 + t$$

$$n=3 \\ (1+t)^4 = 1 \quad ((1+t)^4)^{-1} = 1$$

$$n=7 \\ (1+t)^8 = 1 \quad (1+t)^{-8} = 1$$

$$n=4$$

$$(1+t)^5 = 1 + t + t^4$$

$$\Rightarrow (1+t)^{-5} = 1 + t + t^2 + t^3$$

$$(1+t)^{2k+1} = (1+t)(1+t^{2k}) \\ = 1 + t + t^{2k}$$

$$(1+t)^{-2k-1} = 1 + t + \dots + t^{2k-1}$$

Q Which \mathbb{RP}^n are parallelizable?
 meaning $T\mathbb{RP}^n \cong \underline{\mathbb{R}^n}$ is trivial

Compute $w(T\mathbb{RP}^n)$

If parallelizable must have
 $w_i(T\mathbb{RP}^n) = 0$

Consider $S \rightarrow \mathbb{RP}^n$ tautological bundle

Consider

$$\begin{array}{ccc} S & \hookrightarrow & \mathbb{R}^{n+1} \\ \downarrow & \curvearrowright & \downarrow \\ \mathbb{RP}^n & \hookrightarrow & \mathbb{RP}^n \end{array} \quad \text{tautological injection}$$

$$Q := \mathbb{R}^{n+1}/S$$

$$S^\perp := \left\{ (x, v) \mid \begin{array}{l} x \cdot v = 0 \text{ for } \\ x \in \mathbb{R}\mathbb{P}^n \text{ & } \\ v \in \mathbb{R}^{n+1} \end{array} \right\}$$

We know

$$Q \cong S^\perp$$

$$\text{Note } S \oplus S^\perp \cong \underline{\mathbb{R}^{n+1}}$$

$$\Rightarrow S \oplus Q \cong \underline{\mathbb{R}^{n+1}}$$

CLAIM $T\mathbb{R}\mathbb{P}^n \cong \text{Hom}(S, Q)$ is a canonical isomorphism

Proof: in $\mathbb{R}\text{-top}$ for $\mathbb{R}\mathbb{P}^n$

Choose a unit vector \tilde{x} on L

$$S^n \xrightarrow{?} \mathbb{R}\mathbb{P}^n$$

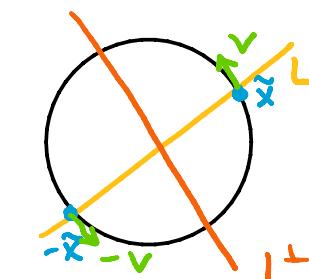
$$\tilde{x} \mapsto \sum \tilde{x}, -\tilde{x}$$

This is a covering map & isomorphism on tangent spaces.

$$L^\perp \cong T_{\tilde{x}} S^n \xrightarrow{T_\varphi} T_{\tilde{x}} \mathbb{R}\mathbb{P}^n \quad \mathbb{R}\mathbb{P}^n \cong TS^n / (\tilde{x}, v) \sim (-\tilde{x}, -v)$$

$$\begin{array}{c} v \mapsto v + \tilde{x} \\ \downarrow \\ v \mapsto -v \end{array}$$

$$\begin{array}{c} v - \tilde{x} \mapsto \\ \downarrow \\ v \end{array}$$



$$\text{Hom}(S, Q) \leftarrow T\mathbb{R}\mathbb{P}^n$$

$$\varphi \leftarrow (\tilde{x}, v)$$

where
 $\varphi(\tilde{x}) := v$ well defined b/c if we use
 $(-\tilde{x}, -v)$ $\varphi(-\tilde{x}) = -v$ by linearity

Note:

$$L^\perp \cong T_{\tilde{x}} S^n$$

$$v \mapsto \text{vector beginning at } \tilde{x} \text{ & ending at } \tilde{x} + v.$$

& that
 $\varphi \mapsto (\tilde{x}, \varphi(\tilde{x}))$
is an isomorphism on fibers.

$$\text{Note } S \oplus Q = \underline{\mathbb{R}^{n+1}} \text{ since } Q = S^\perp$$

Apply $\text{Hom}(S, -)$

$$\text{Hom}(S, S) \oplus \text{Hom}(S, Q) \cong \text{Hom}(S, \underline{\mathbb{R}^{n+1}})$$

Claim: $\text{Hom}(S, S) = \underline{\mathbb{R}}$

Given a scalar a $\text{Hom}(S, S) \times \leftarrow \mathbb{R}_x$ isomorphism on f

$$\varphi_a \leftarrow a$$

$$\text{where } \varphi_a(\tilde{x}) = a\tilde{x} \quad \forall \tilde{x} \in L$$

RMK for any rank 1 vector bundle (line bundle)

$$L \rightarrow X$$

$$\text{Hom}(L, L) \cong \underline{\mathbb{R}}$$

So

$$\begin{aligned} \text{Hom}(S, S) \oplus \text{Hom}(S, Q) \\ \cong \text{Hom}(S, \underline{\mathbb{R}}^{n+1}) \\ \Rightarrow \underline{\mathbb{R}} \oplus T\mathbb{RP}^n \cong \text{Hom}(S, \underline{\mathbb{R}}^{n+1}) \end{aligned}$$

$\text{Hom}(S, \underline{\mathbb{R}}^{n+1})$ is the dual of tensor product.

CLAIM $\text{Hom}(S, \underline{\mathbb{R}}) = S$
only for $T\mathbb{RP}^n$ & \mathbb{R} -vector bundles

we define an inner product on S .

x generates line L &
 $w \in L$

$$S_x \xrightarrow{\cong} \text{Hom}(S, \underline{\mathbb{R}})_x$$

$$w \mapsto \varphi(w) = w \cdot w$$

Note that in \mathbb{C} we can't do this:

$S \not\rightarrow \text{Hom}(S, \underline{\mathbb{C}})$
 $S \rightarrow \mathbb{P}^1$ has no global sections
but its dual $\text{Hom}(S, \underline{\mathbb{C}})$ does

Thus

$$\text{Hom}(S, S) \oplus \text{Hom}(S, Q) = \text{Hom}(S, \underline{\mathbb{R}}^{n+1})$$

becomes

$$\begin{aligned} \underline{\mathbb{R}} \oplus T\mathbb{RP}^n &= (\text{Hom}(S, \underline{\mathbb{R}}))^{n+1} \\ &= \underbrace{S \otimes \dots \otimes S}_{n+1 \text{ times}} \\ &= S^{n+1} \end{aligned}$$

$$\Rightarrow \omega(\underline{\mathbb{R}} \oplus T\mathbb{RP}^n) = \omega(S^{n+1})$$

$$\omega(\underline{\mathbb{R}}) \omega(T\mathbb{RP}^n) = \omega(S^{n+1})$$

$$\begin{aligned} \Rightarrow \omega(T\mathbb{RP}^n) &= \omega(S^{n+1}) \\ &= \omega(S)^{n+1} \\ &= (1+t)^{n+1} \end{aligned}$$

Now we can use Pascal's triangle to answer:

for which n is

$$\omega(T\mathbb{RP}^n) = (1+t)^{n+1} = 1?$$

[i.e. for which n is $T\mathbb{RP}^n$ possibly parallelizable]

$$n = 3$$

$$\omega(T\mathbb{RP}^3) = (1+t)^4 = 1$$

$$n = 7$$

$$\omega(T\mathbb{RP}^7) = (1+t)^8 = 1$$