

## Lecture 3

Q Prove or disprove

Let  $V \rightarrow B$ ,  $E \rightarrow B$  be  $\mathbb{R}$  vector bundles on top spaces

Suppose  $E$  is trivial

$$\Rightarrow w_i(V \oplus E) = w_i(V)$$

Recall

$$w_i(E) = 0 \quad \forall i \neq 0, \quad w_0(E) = 1$$

$$\begin{aligned} \therefore w_i(V \oplus E) &= \sum_{k=0}^i w_k(V) w_{i-k}(E) \\ &= w_i(V) \end{aligned}$$

Recall,

THM [Stiefel]-Whitney

Hirzebruch] There are well defined & uniquely defined Stiefel Whitney classes satisfying the following 4 conditions.

(1) To every vector bundle  $V \rightarrow B$ , there are

$w_i(V) \in H^i(B; \mathbb{Z}_2)$  with  $w_0(V) = 1$  and  $w_i(V) = 0$  for all  $i > \text{rank}(V) = \dim_{\mathbb{R}} V_b$

(2) SW classes are natural

Suppose  $E \rightarrow X$  and  $V \rightarrow B$  are vector bundles

Suppose

$$\begin{array}{ccc} E & \xrightarrow{p'} & V \\ \downarrow & \lrcorner & \downarrow \\ X & \xrightarrow{f} & B \end{array}$$

Q If  $S \rightarrow \mathbb{R}\mathbb{P}^n$  is the tautological bundle, what is  $w_i(S)$  for all  $i$ ?

$$w_0(S) = 1 \text{ by axiom (1)}$$

$$w_i(S) = 0 \text{ for } i > 1$$

Note that

$$\begin{array}{ccc} S_{\mathbb{P}^1} & \xrightarrow{\sim} & S_{\mathbb{P}^n} \\ \downarrow & & \downarrow \\ \mathbb{P}^1 & \xhookrightarrow{\sim} & \mathbb{P}^n \end{array}$$

$$\Rightarrow \text{By (2)} \quad w_k(S_{\mathbb{P}^1}) = f^* w_k(S_{\mathbb{P}^n})$$

$$\text{we know that } w_1(S_{\mathbb{P}^1}) = t$$

$$\begin{aligned} & \notin H^1(\mathbb{P}^n; \mathbb{Z}_2) \cong H^1(\mathbb{P}^1; \mathbb{Z}_2) \\ & \cong \langle t \rangle \end{aligned}$$

$$\Rightarrow w_1(S_{\mathbb{P}^n}) = t$$

is commutative in the topological category and

$$f'_x: E_x \xrightarrow{\sim} V_{f(x)} \text{ is } \mathbb{R}\text{ linear}$$

$$\text{then } w_i(E) = f'^* (w_i(V))$$

(3) For  $V \rightarrow B$ ,  $E \rightarrow B$   $\mathbb{R}$  vector bundles with maybe different rank

$$w_i(V \oplus E) = \sum_{k=0}^i w_k(V) w_{i-k}(E)$$

(4) If  $S \rightarrow \mathbb{R}\mathbb{P}^1$  is the tautological bundle

$$w_1(S) = [t] \in H^1(\mathbb{R}\mathbb{P}^1)$$

From now on  $\underline{\mathbb{R}^n}$  denotes the trivial bundle of rank  $n$ :  $B \times \mathbb{R}^n \rightarrow B$

Suppose  $V \oplus E$  is trivial

$\Rightarrow w_i(V \oplus E) = 0$  for  $i > 0$ , we can solve for  $w_i(V)$  in terms of  $w_i(E)$ !

$$\blacksquare \quad 0 = w_1(V \oplus E) = w_1(V) + w_1(E)$$

$$\Rightarrow w_1(V) = -w_1(E)$$

$$\blacksquare \quad 0 = w_2(V \oplus E) = w_2(V) + w_1(V)w_1(E) + w_2(E)$$

$$\Rightarrow w_2(V) = +w_1(E)^2 - w_2(E)$$

Defn Let  $H^*(B; \mathbb{Z}_2)$  be the ring of formal power series

$$a_0 + a_1 t + a_2 t^2 + \dots$$

$$a_i \in H^i(B; \mathbb{Z}_2)$$

addition comes from addition in  $H^*(B; \mathbb{Z}_2)$  & commutative multiplication from the cup product.

$$(a_0 + a_1 t + a_2 t^2 + \dots)(b_0 + b_1 t + b_2 t^2 + \dots) \\ = a_0 b_0 + (a_0 b_1 + b_0 a_1)t + \dots$$

Recall that  $K[[t]]$  allows for infinitely many terms while  $K[t]$  does not.

e.g

$$H^*(\mathbb{RP}^\infty; \mathbb{Z}_2) = \mathbb{Z}_2[[t]]$$

but sometimes we restrict to  $\mathbb{Z}_2[[t]]$

Defn The total Stiefel Whitney class  $w(E)$  is

$$w(E) = w_0(E) + w_1(E) + w_2(E) + \dots$$

$\Rightarrow$  condition (3) becomes

$$w(E \oplus V) = w(E)w(V)$$

**CLAIM:** the collection of all formal power series forms a commutative group under multiplication.

Set  $x = a_0 + a_1 t + a_2 t^2 + \dots$

$\Rightarrow$  inverse given by  $\frac{1}{1-x} = 1 + x + x^2 + \dots$

More specifically if  $m = \sum_{i=1}^k i_j$

$\Rightarrow$  coefficient in  $x^m$

$$\text{of } a_1^{i_1} \dots a_k^{i_k} \text{ is } \binom{m}{i_1, \dots, i_k} = \frac{m!}{i_1! \dots i_k!} t^m (-1)^m$$

$\Rightarrow$  We can solve for  $\omega(E)$  if we have a  $V$  s.t.

$E \oplus V$  is trivial & we know  $\omega(V)$

OR if  $E \oplus V$  not trivial but we know  $\omega(E \oplus V)$

$$\omega(E) = \omega(E \oplus V)(\omega(V))^{-1}$$

Let  $X \hookrightarrow \mathbb{R}^n$  be a  
submfld,  
we have the tangent  
bundle  $TX \rightarrow X$

& the normal bundle

$$N_X \mathbb{R}^n \rightarrow X$$

$$TX \oplus N_X \mathbb{R}^n = \underline{\mathbb{R}^n}$$

$$\Rightarrow \omega(TX) = \omega(N_X \mathbb{R}^n)^{-1}$$

Q what is  $\omega_i(TS^n)$ ?

Recall  $N_{S^n} \mathbb{R}^n = S^n \times \mathbb{R}$

$$\omega(N_{S^n} \mathbb{R}^n) = 1$$

$$\Rightarrow \omega(TS^n) = 1$$

**RMK** SW did not tell us  
that  $S^2$  has nontrivial  
 $TS^2$

**Defn**  $X$  parallelizable  
if  $TX$  trivial

**Q** Which  $\mathbb{R}\mathbb{P}^n$  are  
parallelizable?