Q: Prove or disprove
Let $V \to B$, $E \to B$ be IR vector bundles on top spaces
Suppose $E$ is trivial
$\Rightarrow \omega_i(V \oplus E) = \omega_i(V)$

Recall
$\omega_i(E) = 0 \quad \forall \ i \neq 0, \ \omega_0(E) = 1$
$\& \ \omega_i(V \oplus E) = \sum_{k=0}^{i} \omega_k(V) \omega_{i-k}(E) = \omega_i(V)$

Recall,
THM [Stiefel-Whitney Hirzbruch] There are well well defined & uniquely defined Stiefel Whitney classes satisfying the following 4 conditions:
(1) To every vector bundle $V \to B$, there are $\omega_i(V) \in H^i(B; \mathbb{Z}_2)$ with $\omega_0(V) = 1$ and $\omega_i(V) = 0$ for all $i > \text{rank}(V) = \dim_{\mathbb{R}} V$

(2) SW classes are natural
Suppose $E \to X$ and $V \to B$ are vector bundles
Suppose $E \to V \xrightarrow{\phi} X \to B$

Q: If $S \to \mathbb{RP}^n$ is the tautological bundle, what is $\omega_i(S)$ for all $i$?
$\omega_0(S) = 1$ by axiom (1)
$\omega_i(S) = 0$ for $i > 1$

Note that
$S_{\mathbb{P}^1} \xrightarrow{\sim} S_{\mathbb{P}^n}$
$\downarrow \quad \downarrow$
$\mathbb{P}^1 \xrightarrow{\sim} \mathbb{P}^n$

$\Rightarrow$ By (2) $\omega_k(S_{\mathbb{P}^1}) = 
\ast^* \omega_k(S_{\mathbb{P}^n})$

We know that $\omega_1(S_{\mathbb{P}^1}) = t$
$\& \ H^1(\mathbb{P}^n, \mathbb{Z}_2) \cong H^1(\mathbb{P}^1, \mathbb{Z}_2)
\Rightarrow \omega_1(S_{\mathbb{P}^n}) = t$

is commutative in the topological category and $f \in : E_x \to V f(x)$ is R-linear
then $\omega_i(E) = f^*(\omega_i(V)$

(3) For $V \to B, E \to B$ IR vector bundles with maybe different rank
$\omega_i(V \oplus E) = \sum_{k=0}^{i} \omega_k(V) \omega_{i-k}(E)$

(4) If $S \to \mathbb{RP}^1$ is the tautological bundle
$\omega_1(S) = [t] \in H^1(\mathbb{RP}^1)$
From now on \( \mathbb{R}^n \) denotes the trivial bundle of rank \( n \) \( \mathbb{R}^n \rightarrow \mathbb{R} \).

Suppose \( V \oplus E \) is trivial, we can solve for \( w_i(V) \).

\[ w_i(V \oplus E) = 0 \quad \text{for} \quad i > 0, \]

in terms of \( w_i(E) \).

- \[ 0 = w_1(V \oplus E) = w_1(V) + w_1(E) \]
- \[ 0 = w_2(V \oplus E) = w_2(V) + w_1(V)w_1(E) + w_2(E) \]
- \[ 0 = w_2(V) = + w_1(E)^2 - w_2(E) \]

**Defn** Let \( H^*(\mathbb{R}; \mathbb{Z}/2) \) be the ring of formal power series

\[ a_0 + a_1t + a_2t^2 + \ldots \]

\[ a_i \in H^i(\mathbb{R}; \mathbb{Z}/2) \]

addition comes from addition in \( H^*(\mathbb{R}; \mathbb{Z}/2) \) & commutative multiplication from the cup product.

\[ (a_0 + a_1t + a_2t^2 + \ldots)(b_0 + b_1t + b_2t^2 + \ldots) \]

\[ = a_0b_0 + (a_0b_1 + b_0a_1)t + \ldots \]

Recall that \( k[[t]] \) allows for infinitely many terms while \( k[t] \) does not.

E.g.

\[ H^*(\mathbb{R}P^\infty; \mathbb{Z}/2) = \mathbb{Z}/2[[t]] \]

but sometimes we restrict to \( \mathbb{Z}/2[t] \).

**Defn** The total Stiefel Whitney class \( w(E) \) is

\[ w(E) = w_0(E) + w_1(E)t + w_2(E)t^2 + \ldots \]

\( \Rightarrow \) condition (3) becomes

\[ w(E \oplus V) = w(E)w(V) \]

**Claim:** the collection of all formal power series forms a commutative group under multiplication.
Set \( x = a_0 + a_1 t + a_2 t^2 + \ldots \)
⇒ inverse given by \( \frac{1}{1-x} = 1 + x + x^2 + \ldots \)
More specifically if \( m = \sum_{i=1}^{k} i_i \)
⇒ coefficient in \( x^m \) of \( a_1^{i_1} \cdots a_k^{i_k} \) is \( \binom{m}{i_1, \ldots, i_k} = \frac{m!}{i_1! \cdots i_k!} \times (-1)^{m-i_1-\cdots-i_k} \)

⇒ We can solve for \( w(E) \) if we have a \( V \) s.t.
  \( E \oplus V \) is trivial & we know \( w(V) \)
  OR if \( E \oplus V \) not trivial but we know \( w(E \oplus V) \)
  \( w(E) = w(E \oplus V) (w(V))^{-1} \)

Let \( X \subseteq \mathbb{R}^n \) be a submfd,
we have the tangent bundle \( TX \to X \)
& the normal bundle \( N_X \mathbb{R}^n \to X \)
⇒ \( TX \oplus N_X \mathbb{R}^n = \mathbb{R}^n \)
⇒ \( w(TX) = w(N_X \mathbb{R}^n)^{-1} \)

\( \Rightarrow \) what is \( w(S^n) \)?
Recall \( N_{S^n} \mathbb{R}^n = S^n \times \mathbb{R} \)
⇒ \( w(N_{S^n} \mathbb{R}^n) = 1 \)
⇒ \( w(TS^n) = 1 \)

RMK SW did not tell us that \( S^2 \) has nontrivial \( TS^2 \)

Defn \( X \) parallelizable if \( TX \) trivial

Q Which \( \mathbb{R} \mathbb{P}^n \) are parallelizable?