

Lecture 2

Q: What do you know about $H^*(\mathbb{R}P^n; \mathbb{Z}_2)$?
 Q: What is the tautological bundle $S \rightarrow \mathbb{P}^n$?

$S \xrightarrow{p} \mathbb{P}^n$
 $p^{-1}(CL) = L$ is the tautological bundle.
 $H^*(\mathbb{R}P^n; \mathbb{Z}_2) = \mathbb{Z}_2[t]/\langle t^{n+1} \rangle$
 How do you obtain this?
 Recall that to build a chain complex we can use the cellular decomposition of $\mathbb{R}P^n$

i.e. we can build $\mathbb{R}P^n$ from disks D^i called cells as follows
 $\mathbb{R}P^n = \mathbb{R}P^{n-1} \cup D^n$
 $\mathbb{R}P^n =$ lines through 0 in \mathbb{R}^{n+1}
 line \mapsto intersection with S^n
 $\mathbb{R}P^n = S^n / \sim -v$

EX:
 $\mathbb{R}P^1 = S^1 / \sim -v$ $a \sim b$ 0 cell

 $\partial(a \sim b) = a - b = 0$ 1 cell
 So $0 \leftarrow C_0 \xrightarrow{\partial} C_1 \leftarrow 0$
 $0 \leftarrow \mathbb{Z}_2 \xrightarrow{\partial} \mathbb{Z}_2 \leftarrow 0$
 $\Rightarrow H_*(\mathbb{R}P^1) = \begin{cases} \mathbb{Z}_2 & * = 0, 1 \\ 0 & \text{otherwise} \end{cases}$

For $\mathbb{R}P^2$ instead, 
 $\mathbb{R}P^2 = \mathbb{R}P^1 \cup D^2$
 $0 \leftarrow \mathbb{Z}_2 \xrightarrow{\partial} \mathbb{Z}_2 \xrightarrow{\partial} \mathbb{Z}_2 \leftarrow 0$
 $H_i(X; \mathbb{Z}_2) = \ker \partial / \text{Im } \partial$ in deg i of C_*
 $H_i(X; \mathbb{Z}_2) = \ker \delta / \text{Im } \delta$ in degree i of $\text{Hom}(C_*(X); \mathbb{Z}_2)$

$t =$ class of hyperplane $\mathbb{R}P^{n-1}$ obtained from $\mathbb{R}P^{n-1} \hookrightarrow \mathbb{R}P^n$
 wonderful Poincaré duality gives us for compact, no boundary manifolds X
 $H_n(X; \mathbb{Z}_2) \cong \mathbb{Z}_2$
 fundamental class $[t] \leftrightarrow 1$

Furthermore, there is an isomorphism
 $H^i(X; \mathbb{Z}_2) \xrightarrow{\cong} H_{n-i}(X; \mathbb{Z}_2) \cap [X]$
 $n = \dim X$

t is the image in $H^1(\mathbb{R}P^n; \mathbb{Z}_2)$

from

$$\mathbb{Z}_2 \cong H_{n-1}(\mathbb{R}P^{n-1}; \mathbb{Z}_2) \rightarrow H_{n-1}(\mathbb{R}P^{n-1}, \mathbb{Z}_2) \xrightarrow{\cong} H^1(\mathbb{R}P^n; \mathbb{Z}_2)$$

$$[t] \longmapsto [t]$$

" t is the class of the hyperplane $\mathbb{R}P^{n-1}$ "

Slogan: 'submanifolds give cohomology classes'
'cup product correspond to intersection'

$t^2 = t \cup t =$ intersection of
 $\begin{matrix} \mathbb{R}P^n \\ H^2 \end{matrix}$ $\begin{matrix} \mathbb{R}P^n \\ H^1 \end{matrix}$ hyperplane w/
 itself after moving
 one hyperplane so
 its transverse.

$t^i \neq 0$ b/c i linear eqns determines
 $\mathbb{P}(\mathbb{R}^{n+1-i}) = \mathbb{R}P^{n-i}$ in
 $\mathbb{R}P^n$
 which is nonempty for $i < n+1$
 But its empty for $t^{n+1} = 0$ b/c $\mathbb{P}^n = \emptyset$

Stiefel Whitney classes

for topological \mathbb{R}^n bundles
 [Milnor Stasheff]

for each real vector bundle
 $V \rightarrow B$ associate Stiefel
 Whitney classes w_i such that
 in $H^i(B; \mathbb{Z}_2)$ for $i=0, 1, \dots$

Motivation

(1) w_1 tells us if V orientable

$$w_1(V) = 0 \iff V \text{ orientable}$$

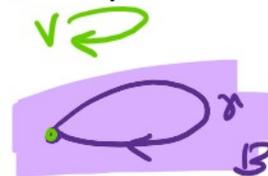
More specifically,

$$w_1(V) \in H^1(B; \mathbb{Z}_2)$$

$$\cong \text{Hom}(H_1(B, \mathbb{Z}_2), \mathbb{Z}_2)$$

universal coefficient thm

γ loop in $B \longmapsto [\gamma]$ in
 $H_1(B; \mathbb{Z}_2)$



$$w_1(V)[\gamma] = 0 \iff \forall \gamma \rightarrow \gamma \text{ orientable}$$

e.g Möbius band

$$\mathbb{R}/\gamma \rightarrow \gamma^1$$

non orientable
 $w_1(V)[\gamma] \neq 0$



(2) $w_i(V)$ is an obstruction to
 the existence of $n+i-1$
 linearly independent
 sections of V on i -skeleton of B

e.g. $W_n(V)$ is an obstruction to having a global nowhere vanishing section.

(3) SW numbers are i_1, \dots, i_m , for all m s.t.

$$\sum_j i_j = n, \text{ so}$$

$$w_{i_1} w_{i_2} \dots w_{i_m} \in H^n(B; \mathbb{Z}_2)$$

$$\cong H_0(B; \mathbb{Z}_2)$$

$$\cong \mathbb{Z}_2$$

THM [Thom] Two mflds are cobordant \Leftrightarrow SW numbers of their tangent bundles are the same

Aside:

Given vector bundles $E \rightarrow B, V \rightarrow B$, for $b \in B$ we have E_b, V_b are \mathbb{R} -vector spaces,

the following are also vector spaces:

$$\Lambda^2 E_b, E_b \oplus V_b, E_b \otimes V_b, \text{Hom}(E_b, V_b)$$

These vector space operations give rise to the following vector bundles

$$\Lambda^2 E, E \oplus V, E \otimes V,$$

$$\text{Hom}(E, V)$$

see Milnor-Stasheff

THM [Stiefel-Whitney Hirzebruch] There are well defined & uniquely defined Stiefel Whitney classes satisfying the following 4 conditions.

(1) To every vector bundle $V \rightarrow B$, there are $w_i(V) \in H^i(B; \mathbb{Z}_2)$ with $w_0(V) = 1$ and $w_i(V) = 0$ for all $i > \text{rank}(V) = \dim_{\mathbb{R}} V_b$

(2) SW classes are natural. Suppose $E \rightarrow X$ and $V \rightarrow B$ are vector bundles

Suppose

$$\begin{array}{ccc} E & \xrightarrow{f'} & V \\ \downarrow \cong & & \downarrow \\ X & \xrightarrow{f} & B \end{array}$$

is commutative in the topological category and

$$f'_x: E_x \cong V_{f(x)} \text{ is } \mathbb{R}\text{-linear}$$

$$\text{then } w_i(E) = f'^*(w_i(V))$$

(3) For $V \rightarrow B, E \rightarrow B$ \mathbb{R} vector bundles with maybe different rank

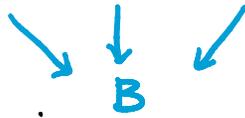
$$w_i(V \oplus E) = \sum_{k=0}^i w_k(V) w_{i-k}(E)$$

(4) If $S \rightarrow \mathbb{R}P^1$ is the tautological bundle

$$w_1(S) = [1] \in H^1(\mathbb{R}P^1)$$

Q What about short exact sequences of vector bundles?

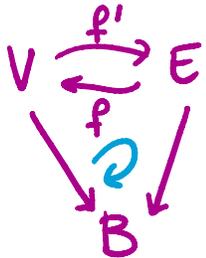
In Top $E \rightarrow V \rightarrow V/E$ splits



$$\Rightarrow \omega(V) = \sum_{k=0}^i \omega_k(E) \omega_{i-k}(V/E)$$

CLAIM: If $V \rightarrow B$ and $E \rightarrow B$ are isomorphic vector bundles

i.e



isomorphism
fiberwise

$$f'f = \text{Id}_V$$

$$ff' = \text{Id}_E$$

$$\Rightarrow \omega_i(V) = \omega_i(E)$$

Proof: Follows directly from (2)

CLAIM: If $V = B \times \mathbb{R}^n \rightarrow B$ is the trivial vector bundle

$$\Rightarrow \omega_i(V) = 0 \quad \forall i > 0.$$

Proof: First note that

$$B \times \mathbb{R}^n = (B \times \mathbb{R}^{n-1}) \oplus (B \times \mathbb{R})$$

$$\Rightarrow \omega_i(B \times \mathbb{R}^n) = \sum_{k=0}^i \omega_k(B \times \mathbb{R}^{n-1}) \omega_{i-k}(B \times \mathbb{R})$$

by (3) it suffices to show $\omega_i(B \times \mathbb{R}) = 0$ for all i .

Since $\omega_{i-k}(B \times \mathbb{R}) = 0$ for all $i-k > 1$

and $\omega_0(B \times \mathbb{R}) = 0$ by (1) we only need to check $\omega_1(B \times \mathbb{R})$

Choose a map $B \xrightarrow{f} \mathbb{R}P^1$, $f(b) = p$ point in $\mathbb{R}P^1$

$$B \times \mathbb{R} \xrightarrow{F} S$$

$$F(b, l) = (p, l) \quad \& \text{ using (2) to}$$

$$\downarrow \quad \downarrow$$

$$B \xrightarrow{f} \mathbb{R}P^1$$

pullback

$$0 = f^*(\omega_1(S)) = \omega_1(B \times \mathbb{R})$$