

Lecture 16

Tubular nhd thm:

Let $Y \hookrightarrow X$ be an embedding of smooth manifolds. Then \exists an open nhd of Y and $U \cong_{\text{diffeo}} N_Y X$

$\curvearrowleft Y \nearrow \text{zero section}$

Proof: $\exp : T_Y X \rightarrow X$

THM (Purity thm)

Given $Y \hookrightarrow X$ embedding of smooth mfds

$$\text{Th}(N_Y X) = X/X - Y$$

$$\underline{\text{Pf: }} X/X - Y = U/U - Y$$

$$= \frac{N_Y X}{N_Y X - Z(Y)}$$

$$\simeq \text{Th}(N_Y X)$$

Alternatively (MacPherson) consider deformations to the normal bundle

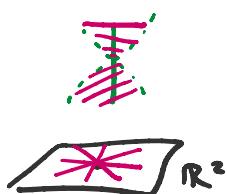
Blowups $\text{Bl}_Y X$ for real manifolds

$$\text{Bl}_Y X = X - U / \sim$$

$\partial(X - U) = S(N_Y X)$
sphere bundle attached to $\mathbb{P}N_Y X$

$$\begin{array}{ccc} \text{Bl}_Y & \rightarrow & X \\ \parallel & \nearrow & \downarrow \\ \mathbb{P}N_Y X \times_{X-Y} & \xrightarrow{\quad} & Y \end{array} \quad \text{where } v \sim -v.$$

Example: $\text{Bl}_0 \mathbb{R}^2 = \mathcal{O}_{\mathbb{RP}^1(-1)}$



replace a point with the space of complex lines passing through that point

$$\{ (x,y) \times (X,Y) \mid \begin{matrix} (x,y) \text{ is on } \{ (x,y) \mid t(x,y) = 0 \} \\ \text{the line } \{ (x,y) \mid t(x,y) = 0 \} \end{matrix} \} / \mathbb{R}$$

$$\mathcal{O}_{\mathbb{RP}^1(-1)} //$$

Ex $\text{Bl}_0 \mathbb{R} = \mathbb{R}$

Let

$$D_Y X = \text{Bl}_{Y \times 0}(X \times \mathbb{R}) - \text{Bl}_{Y \times 0}(X \times 0)$$

$$\begin{array}{ccc} \pi \downarrow & & \downarrow \\ \mathbb{R} & \longleftarrow & Y \times \mathbb{R} \end{array}$$

Q Can you identify $D_Y X$ with the total space of some bundle?

$$D_0 \mathbb{R} = \text{Bl}_{0 \times 0}(\mathbb{R} \times \mathbb{R})$$

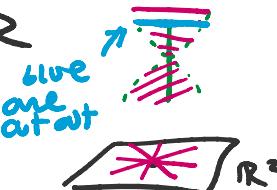
$$- \text{Bl}_{0 \times 0}(\mathbb{R} \times 0)$$

$$= \text{Bl}_0 \mathbb{R}^2 - \text{Bl}_0 \mathbb{R}$$

$$= \mathcal{O}_{\mathbb{RP}^1(-1)} - \mathbb{R}$$

$$= \mathcal{O}_{\mathbb{RP}^1(-1)}|_{\mathbb{R}}$$

$$\simeq \mathbb{R} \times \mathbb{R}$$



In $\pi^{-1}(1)$ we have $y \hookrightarrow X$
but at $\pi^{-1}(0)$ we have $y \hookrightarrow N_y X$
zero section
 $N_y X = \text{IP}(N_{y,x_0}(X \times \mathbb{R})) - \text{IP}(N_y X)$
 $= \text{IP}(N_y X \oplus \underline{\mathbb{R}}) - \text{IP}(N_y X)$
↑
trivial \mathbb{R}
bundle of rank 1

Then $\frac{D_y X}{D_y X - Y \times \mathbb{R}}$ induced by inclusion of $\pi^{-1}(1)$ $\xleftrightarrow{\pi^{-1}(1)} \frac{X}{X - Y}$

\uparrow induced by inclusion of $\frac{N_y X}{N_y X - Z(Y)}$.

One can show that they are weak equivalences
 $\Rightarrow Th(N_y X) \leftrightarrow Th(X)$
canonical map \square

(the construction Grothendieck missed)

$$H^{d-n}(Th(N_y X)) \cong H^{d-n}(X/X - Y) \xrightarrow{\cong} H^*(X)$$

$\overset{u^*}{\leftarrow} \qquad \qquad \qquad \overset{v^*}{\rightarrow}$

Thom class

Defn u^* is the dual cohomology class to y .
Let $\Delta: X \rightarrow X \times X$ be the diagonal. Let v^* be the dual class to $\Delta(X)$.
 H^* will have field coefficients.

Last time:
 $y \hookrightarrow X$ embedding for manifolds Y, X w/ $\dim Y = n, \dim X = d$
 $\Rightarrow N_y X$ is oriented

THM

$$u^* = \sum_{i=1}^r (-1)^{\dim b_i} \pi_1^* b_i \cup \pi_2^* c_i$$

where $\{b_1, \dots, b_r\} \subset H^*(X)$ is a basis and $c_i \in H^{n-\dim b_i}(X)$ s.t.

$$(b_i \cup c_j)[x] = \delta_{ij}$$

$$\begin{matrix} X \times X \\ \pi_1 \downarrow \qquad \qquad \qquad \downarrow \pi_2 \\ X \qquad \qquad \qquad X \end{matrix}$$

Our proof from last time was done except we NTS the following lemma

LEMMA $\forall a \in H^*(X)$

$$\pi_1^* a \cup u' = \pi_2^* a \cup u'$$

u' is supported on the diagonal on which π_1 and π_2 are the same map

$$\pi_1|_{\Delta(X)} = \pi_2|_{\Delta(X)}$$

\Rightarrow for U a tubular nbhd of $\Delta(X)$

$$\pi_1|_U \simeq \pi_2|_U \text{ homotopic}$$

u' is in the image of $H^{d-n}(U) \rightarrow H^{d-n}(X)$.

From lecture 14:

PROPOSITION Let V be a rank n \mathbb{C} -vector bundle

$$\Rightarrow e(V) = c_n(V)$$

$$\text{NTS } e(\mathcal{O}_{\mathbb{CP}^1}(-1)) = -t$$

$t \in H^2(\mathbb{CP}^1; \mathbb{Z})$ is positively oriented generator.

Proof: Let V be a \mathbb{C} vector bundle on a paracompact topological space.

$\Rightarrow V$ admits a hermitian metric

$\Rightarrow V^* \text{Hom}(V, \mathbb{C}) \cong \bar{V}$
conjugate bundle

the \mathbb{C} vector bundle w/ conjugate \mathbb{C} -multiplication

$\Rightarrow \bar{V}$ is V with opposite orientation.

$$\Rightarrow e(V^*) = e(\bar{V}) = (-1)^{rk(V)} e(V)$$

\Rightarrow it suffices to show $e(\mathcal{O}_{\mathbb{CP}^1}(1)) = t$

CLAIM $\pi_1(\mathcal{O}_{\mathbb{CP}^1}(1)) = \mathbb{CP}^2$

$$\mathbb{P}^1 \hookrightarrow \mathbb{P}^2$$

$$[x, y] \mapsto [0, x, y]$$

$$\mathbb{P}^2 - [1, 0, 0] \xrightarrow{\pi_1} \mathbb{P}^1 \text{ normal bundle } N_{\mathbb{P}^1} \mathbb{P}^2 \rightarrow \mathbb{P}^1$$

$$[z, x, y] \mapsto [x, y]$$

fiber over $[x_0, y_0]$ is the line $\{[z, x, y] \mid \begin{matrix} z=x_0 \\ z \in \mathbb{C} \end{matrix}, y=y_0\}$

$$\begin{aligned} [t, x, y] &\leftrightarrow (x, y) \times t \\ N_{\mathbb{P}^1} \mathbb{P}^2 &\cong \mathcal{O}_{\mathbb{P}^1}(1) \\ &\quad \| \\ &\quad \frac{\mathbb{C}^2 - \{0\} \times \mathbb{C}}{(x, y) \times t \sim (\lambda x, \lambda y) \times \lambda t} \\ &\quad \lambda \in \mathbb{C}^* \end{aligned}$$

\Rightarrow By the purity theorem
 $T_h(\mathcal{O}_{\mathbb{P}^1}(1)) = T_h(N_{\mathbb{P}^1} \mathbb{P}^2)$
 $= \mathbb{P}^2 / \mathbb{P}^2 - \mathbb{P}^1 \cong \mathbb{P}^2$

We claim $t \in H^2(T_h(\mathcal{O}_{\mathbb{P}^1}(1))) \cong H^2(\mathbb{CP}^2)$
is the Thom class $\mathcal{O}_{\mathbb{P}^1}(1)$
NTS $t|_{T_h(\mathcal{O}(1)_{[x, y]})}$ is
a positive generator
the inclusion
 $T_h(\mathcal{O}(+1)_{[x, y]}) \rightarrow T_h(\mathcal{O}(1))$
 $\mathbb{P}^1 \xrightarrow{\text{line through } 0 \text{ to } [x, y]} \mathbb{P}^2$

Then,

$$H^+(\mathbb{CP}^2) \rightarrow H^+(\mathbb{CP}^1)$$

$$t \longrightarrow t$$

□

Pf:
 $T_h(TX) \cong T_h(N_{\Delta(X)}(X \times X)) \leftarrow X$
 u Thom class in
 $H^{\dim X}(T_h(TX))$ and it
maps to u' in $H^+(X \times X)$
 $e(X)[X] = z^* u[X] = \Delta^* u'[X]$

Euler class

Claim X smooth compact,
oriented mfld

$$e(TX)[X] = \sum_{i=1}^{\dim X} (-1)^i \dim_{\mathbb{Q}} H^i(X; \mathbb{Q})$$

$$= \chi(X)$$

$$\begin{aligned} u' &= \sum (-1)^{\dim b_i} \pi_1^* b_i \cup \pi_2^* b_i^* \\ \Delta^* u'[X] &= \sum_{i=1}^r (-1)^{\dim b_i} b_i \cup b_i^*(TX) \\ &= \sum_{i=1}^r (-1)^{\dim b_i} \\ &= \sum_{j=1}^{\dim(X)} (-1)^j \operatorname{rk}(H^j) \\ &= \chi(X) \end{aligned}$$