

Lecture 15

Suppose we have an embedding of smooth manifolds

THM $X/x-y \cong Tn(N_x Y)$
 canonical homotopy
 equivalence where
 $N_x Y$ is the normal
 bundle of $Y \hookrightarrow X$

COROLLARY

$$H^*(X, X-Y; \mathbb{Z}) \cong H^*(Th(N_Y X); \mathbb{Z})$$

Suppose $\dim X = d \neq$

dim $Y = n \Rightarrow$ rank of $N_x Y = d - n$
 If $N_x Y$ is oriented, there exists a
 Thom class $u \in H^{d-n}(N_x Y)$
 and

$$H^{d-n}(T_h(N_Y X)) \cong H^{d-n}(X, X-Y) \xrightarrow{\text{restriction}} H^{d-n}(X) \rightarrow H^{d-n}(Y) \xrightarrow{\text{restriction}}$$

$\mathcal{U} \longleftarrow \mathcal{U}_Y \longrightarrow Z^*(\mathcal{U})$

\mathcal{U}_Y is the dual cohomology class to Y . $Z^*(\mathcal{U}) \cong e(N_Y X)$

RMK we will see that

$$N_Y X \simeq \cup_{C \in X} \text{tubular nbhd of } Y$$

Let $\Delta : X \rightarrow X \times X$ denote the diagonal

LEMMA $N_x(X \times X) \cong TX$

Proof: $N_x(X \times X) \subset T(X \times X)$

$$T(X \times X) \cong TX \times TX$$

If $v, w \in T_p X$ for $p \in X$

then $(\omega, v) \in T(X \times X)_{(p, p)}$

(w, v) is tangent to

$$\Delta X \Leftrightarrow w = v$$

while (w, v) is normal
 $\leftarrow \wedge x \Leftrightarrow w = -v$

so we have a map

$$Tx \rightarrow N_x(x \times x) \subset T(x \times X) \Big|_{\Delta x}$$

$$T_{X_p} N_x(x) \big|_{(p,p)}$$

$$\tilde{\omega} \mapsto (\omega, -\omega)$$

which is a linear iso
of vector bundles

Defn Let X be an n -dim mfd, we say that X is oriented if TX is oriented.
 $\Leftrightarrow \exists$ a choice of generator of $H^n(T_p X, T_p X - \{p\})$ for each $p \in X$, which varies conts in p .

That is, $\forall p, \exists$ an open nhd of p and $\gamma \in H^n(TX|_U, TX|_U - Z)$ which restrict to each chosen generator

Using weak canonical

$$T_p X / T_p X - Z \cong X/X - p$$

Z zero section
This is equivalent to saying that for each $p \in X \exists$ a chosen

generator of $H^n(X, X - p)$ that varies continuously with p .

This gives a dual generator of $H^n(X, X - p)$.

THM [see Hatcher PD]
Given X compact, oriented, $\exists [x] \in H_n(X)$ st. $[x] \in H_n(X, X - p)$ is the chosen generator.

Since X is oriented, we have a Thom class

$$u \in H^n(X \times X, X \times X - \Delta(X))$$

$$\downarrow \quad \downarrow \\ u' \in H^n(X \times X)$$

"dual to the diagonal of X "

u' records intersections with Z , the zero section.

Slant product:
use field coefficients for simplicity

$$H^{p+q}(X \times Y) \otimes H_q(Y) \rightarrow H^p(X) \\ \alpha \otimes \beta \mapsto \alpha \beta$$

construction: using the Künneth formula.

$$H^*(X \times Y) \cong H^*(X) \otimes H^*(Y)$$

given $a \in H^*(X)$ and $b \in H^*(Y)$

$$a \otimes b \otimes \beta \mapsto a(b(\beta)) = a \cdot b \cdot \beta$$

$$Ex \quad u' / [x] = 1$$

$$H^*(X \times X) \xrightarrow{/[x]} H^*(X)$$

$$p \in X$$

$$i_x : X \rightarrow X \times X$$

$$i_x(x_0) = (p, x_0)$$

$$\Rightarrow H^*(X \times X) \xrightarrow{/[x]} H^*(X)$$

$$\downarrow \text{restrict} \quad \downarrow \text{restrict} \\ H^*(p \times X) \cong H^*(X) \xrightarrow{/[x]} H^*(\{p\})$$

Then,

$$H^n(X \times X) \xrightarrow{[1 \times X]} H^0(X)$$

$$\downarrow \quad \quad \quad \downarrow$$

$$H^n(p \times X) \cong H^n(X) \xrightarrow{[X]} H^0(\{p\})$$

Assume X is connected, it is sufficient to show that the image of $[X]$ in $H^0(\{p\})$ is 1.

$$i_x^*(u') = 1 \times (i_x^*(u'))$$

$$\in H^n(X \times X, X \times X - \Delta(X))$$

$$H^n(X \times X, X \times X - \Delta(X))$$

$$\downarrow$$

$$H^n(p \times X, p \times X - p \times p)$$

$$\downarrow$$

$$H^n(X, X - p)$$

By construction our chosen generator of $H^n(X, X - p)$ is $i_x^*(u')$

\Rightarrow By defn of $[X]$

$$i_x^*(u') / [X] = 1$$

Choosing a $p \in X$ for each connected component, we are done.

THM [Poincaré Duality]

X compact oriented n dim mfld
 \mathbb{K} is the field of coefficients.

The cup product

$$H^k(X; \mathbb{K}) \otimes H^{n-k}(X; \mathbb{K}) \xrightarrow{\langle -, [X] \rangle} H^n(X) \rightarrow \mathbb{K}$$

induces a perfect pairing.

i.e. the induced map

$$H^k(X) \xrightarrow{\cong} \text{Hom}(H^{n-k}(X), \mathbb{K})$$

is an isomorphism.

PROPOSITION: Let b_1, \dots, b_r be a basis of $H^k(X)$ and let b_1^*, \dots, b_r^* be the dual basis s.t. $b_j^* \cdot b_i = \langle b_i, b_j^* \rangle = \delta_{ij} = \begin{cases} 1 & i=j \\ 0 & i \neq j \end{cases}$

$$\Rightarrow u' = \sum_{i=1}^r (-1)^{\dim(b_i)} (b_i \times b_i^*)$$

Let $a \in H^k(X)$

$$1 \times a, a \times 1 \in H^k(X \times X)$$

By invariance of u' under swapping X

factors in $X \times X$:

$$(1 \times a) \cup u' = (a \times 1) \cup u'$$

(this last sentence is)
 proved in lecture 16

Let b_1, \dots, b_r be a basis of $H^k(X)$.

We can write

$$u' = \sum_{i=1}^r b_i \times c_i$$

$c_i \in H^{n-\dim(b_i)}(X)$. Let $a \in H^k(X)$

$$1 \times a, a \times 1 \in H^k(X \times X)$$

$$\Rightarrow (1 \times a) \cup u' / [X]$$

$$= (1 \times a) \cup \sum b_i \times c_i / [X]$$

$$= \sum (-1)^{\dim a \cdot \dim b_i} (b_i \times a \cdot c_i) / [X]$$

$$= \sum_{b_i \in \{a_i c_i, [x]\}} (-1)^{\dim a \cdot \dim b_i}$$

If we let $a = b_j$

$$b_j = \sum_{b_i \in \{b_j c_i, [x]\}} (-1)^{\dim b_j \cdot \dim b_i} b_i$$

Since the b_i 's are a basis

$$\langle b_j c_i, [x] \rangle = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases}$$

$\Rightarrow c_i$ dual basis of b_i

$$c_i = (-1)^{\dim b_i} b_i^*$$