

Lecture 14

Let $V \rightarrow B$ be a \mathbb{R} -vector bundle
 $T_h(V) = D(V)/S(V)$ if V has a Euclidean metric
 $\simeq V/V - \mathcal{Z}(B)$
 $\simeq \mathbb{P}(V \otimes \mathbb{R})/\mathbb{P}(V)$

where $\mathcal{Z}: B \rightarrow V$ is the zero section.

Given another vector bundle $V' \rightarrow B'$, show that

$$T_h(V') \wedge T_h(V) \simeq T_h(V \times V')$$

where $V \times V' \rightarrow B \times B'$

We have

$$V \otimes V' \rightarrow B \times B'$$

given by

$$\begin{array}{ccc} V \otimes V' & \downarrow & \\ V & \searrow & B \times B' & \swarrow & V' \\ & B & & B' & \end{array}$$

$$\frac{D(V \times V')}{S(V \times V')} = \frac{D(V) \times D(V')}{\partial(D(V) \times D(V'))}$$

$$= \frac{D(V) \times D(V')}{(\partial(V) \times \partial(V')) \cup (\partial(D(V)) \times V')}$$

$$\hookrightarrow \frac{D(V)}{S(V)} \times \frac{D(V')}{S(V')} \\ \{ \# \times \frac{D(V')}{S(V')} \cup \frac{D(V)}{S(V)} \times \# \}$$

$$= \frac{T_h(V) \times T_h(V')}{\{ \# \times T_h(V') \cup T_h(V) \times \# \}}$$

$$= T_h(V) \wedge T_h(V')$$

Map above takes $(a, b) \in D(V) \times D(V')$ to $(\bar{a}, \bar{b}) \in T_h(V) \times T_h(V')$

Recall that for a rank n vector bundle $V \rightarrow B$, it has an orientation if $\exists \alpha \in H^n(T_h(V); \mathbb{Z})$ s.t.

$$\alpha|_{T_h(V_b)} \in H^n(T_h(V_b); \mathbb{Z}) \simeq \mathbb{Z}^n$$

is a generator

The euler class, $e(V) := \mathbb{Z}^n \alpha$

EX 1

$\mathcal{O}(-1) \rightarrow \mathbb{CP}^1$ rank 2 oriented oriented \mathbb{R} -bundle

$$\mathbb{CP}^1 \cong S^2$$

$$H^*(\mathbb{CP}^1; \mathbb{Z}) \simeq \mathbb{Z}[+]/(+^2)$$

$$\Rightarrow e(\mathcal{O}(-1)) = -1$$

We will prove this later on.

PROPERTIES OF THE EULER CLASS

□ If we have any map

$$f: B' \rightarrow B$$

$$e(f^* B) = f^* e(B)$$

(Note that pullbacks of oriented bundles are oriented.)

Proof:

Let \mathcal{U}_V be the Thom class of $V \rightarrow B$, and $\mathcal{U}_{f^* V}$ the Thom class of $f^* V \rightarrow B$

Then

$$f^*(\mathcal{U}_V) = \mathcal{U}_{f^* V}$$

□ If we reverse the orientation on each fiber the Euler class changes sign since $\mathcal{U} \rightarrow -\mathcal{U}$.

Claim: If $n = \text{rank}(V)$ is odd
 $\Rightarrow 2e(V) = 0$

Consider $\tilde{V} \rightarrow B$ constructed from V with fiber orientation reversed since V has odd rank

$$e(V) = -e(\tilde{V}).$$

Note that there $\exists \tilde{f}: \tilde{V} \rightarrow V$

$$\begin{array}{ccc} \tilde{V} & \xrightarrow{\tilde{f}} & V \\ \downarrow & & \downarrow \\ B & \xrightarrow{f = \text{Id}} & B \end{array}$$

where \tilde{f} is a linear isomorphism that switches orientation on fibers.

Since

$$e(f^* V) = f^* e(V)$$

$$\begin{aligned} e(\tilde{V}) &= \text{Id}^*(e(V)) \\ &= e(V) \end{aligned}$$

\Rightarrow

$$\begin{aligned} 2e(V) &= e(V) + e(V) \\ &= e(\tilde{V}) - e(\tilde{V}) \\ &= 0 \end{aligned}$$

Claim: Let $V \rightarrow B$, $V' \rightarrow B'$ be \mathbb{R} -topological oriented vector bundles

$$\Rightarrow e(V \times V') = \pi_B^* e(V) \cup \pi_{B'}^* (e(V'))$$

$$\begin{array}{ccc} B \times B' & \xrightarrow{\pi_B} & B \\ & \xrightarrow{\pi_{B'}} & B' \end{array}$$

Pf:

$$\mathcal{U}_{V \times V'} = \pi_B^* \mathcal{U}_V \cup \pi_{B'}^* \mathcal{U}_{V'}$$

b/c $\pi_B^* \mathcal{U}_V|_{(x,x')} \cup \pi_{B'}^* \mathcal{U}_{V'}|_{(x,x')}$
 $(x,x') \in B \times B'$
 is a positively oriented generator

$H^n(S^n) \times H^m(S^m) \rightarrow H^{n+m}(S^{n+m})$
 takes positive generators to positive generators.

$$\begin{aligned} \mathcal{Z}_{V \times V'}^*(U_{V \times V'}) &= \prod_B^* \mathcal{Z}_V^* U_V \cup \prod_B^* \mathcal{Z}_{V'}^* U_{V'} \\ &= \prod_B^* e(V) \cup \prod_B^* e(V') \end{aligned}$$

COROLLARY: $V \rightarrow B, V' \rightarrow B$ be two oriented \mathbb{R} -topological vector bundles
 $e(V \oplus V') = e(V) \cup e(V')$

Pf let

$\Delta: B \rightarrow B \times B$ be the diagonal

$$V \oplus V' = \Delta^*(V \times V') \quad \square$$

$$\text{EX: } \Theta_{\mathbb{C}\mathbb{P}^\infty}(-1) \rightarrow \mathbb{C}\mathbb{P}^\infty$$

$i: \mathbb{C}\mathbb{P}^1 \hookrightarrow \mathbb{C}\mathbb{P}^\infty$ is the inclusion of the 2-skeleton

$$i^*(e(\Theta_{\mathbb{C}\mathbb{P}^\infty}(-1))) = e(i^*(\Theta_{\mathbb{C}\mathbb{P}^\infty}(-1)))$$

$$= e(\Theta_{\mathbb{C}\mathbb{P}^1}(-1))$$

$$= -t$$

i^* is an isomorphism on H^2

$$i^*: H^2(\mathbb{C}\mathbb{P}^\infty) \xrightarrow{\cong} H^2(\mathbb{C}\mathbb{P}^1)$$

$$\mathbb{Z}[t] \rightarrow \mathbb{Z}[t]/t^2$$

is the quotient map.

Q: What is $e(\Theta(-1) \times \dots \times \Theta(-1))$?

$$\pi_1^* \Theta(-1) \oplus \pi_2^* \Theta(-1) \oplus \dots \oplus \pi_n^* \Theta(-1)$$

$$\downarrow$$

$$\underbrace{\mathbb{C}\mathbb{P}^\infty \times \dots \times \mathbb{C}\mathbb{P}^\infty}_{n \text{ times}}$$

$$e(\Theta(-1) \times \dots \times \Theta(-1)) = (-1)^n t_1 v \dots v t_n$$

Claim For $V \rightarrow B$ a \mathbb{C}^n bundle viewed as a \mathbb{R}^{2n} bundle

$$e(V) = C_n(V)$$

Pf: By naturality we may assume V is the canonical bundle

$$S_{\mathbb{C}\text{Gr}_n} \rightarrow \mathbb{C}\text{Gr}_n$$

Let $f: \mathbb{C}\mathbb{P}^\infty \times \dots \times \mathbb{C}\mathbb{P}^\infty \rightarrow \mathbb{C}\text{Gr}_n$ be the map classifying

$$\Theta(-1) \times \dots \times \Theta(-1) \rightarrow \mathbb{C}\mathbb{P}^\infty \times \dots \times \mathbb{C}\mathbb{P}^\infty$$

$$f^*: H^*(\mathbb{C}\text{Gr}) \hookrightarrow H^*(\mathbb{C}\mathbb{P}^\infty \times \dots \times \mathbb{C}\mathbb{P}^\infty)$$

$$\mathbb{Z}[t_1, \dots, t_n] \quad \mathbb{Z}[t_1, \dots, t_n]$$

$t_i \mapsto \pm i$ th elementary sym. polynomial

$$f^* e(S_{\mathbb{C}\text{Gr}_n}) = e(f^* S_{\mathbb{C}\text{Gr}_n})$$

$$= e(\Theta(-1) \times \dots \times \Theta(-1))$$

$$= (-1)^n t_1 v \dots v t_n$$

$$= f^* C_n(S_{\mathbb{C}\text{Gr}_n})$$

Since f is injective

$$f^* e = f^* C_n \Rightarrow e = C_n \quad \square$$

RMK: This is why c_i is defined by
 $t^n - c_1 t^{n-1} + c_2 t^{n-2} - \dots + (-1)^n c_n$
 $\Rightarrow c_n = (-1)^n t_1 \cdots t_n$

Application 2:

Let $V \rightarrow B$ be an oriented
IR bundle w/ $2e(V) \neq 0$

Then V doesn't have an odd
dim subbundle.

Pf: If $V' \subset V$ is an odd
dim subbundle
 $\Rightarrow e(V') = 0$

$$V' \hookrightarrow V$$

$$V \cong V' \oplus (V')^\perp$$

$$e(V) = e(V') \cup e((V')^\perp)$$

$$2e(V) = 2(e(V') \cup e((V')^\perp)) \\ = 0 \quad \Leftarrow$$

Claim $\Theta(-1) \rightarrow \mathbb{CP}^1$

$$e(\Theta(-1)) = -t$$

proof postponed to
lecture 16