

Lecture 13

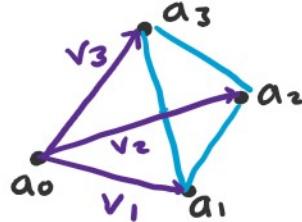
Orientations

Let V be an n -dim \mathbb{R} vector space. Two basis $\{v_1, \dots, v_n\}$ & $\{v'_1, \dots, v'_n\}$ are equivalent if they differ by a matrix $A = (a_{ij})$ with $\det A > 0$

$$v_j = \sum_i a_{ij} v'_i$$

Defn 1: An orientation of V is one of the two equivalence classes of bases.

Given an n -simplex oriented convex hull $\{a_0, \dots, a_n\} \subset V$ we have an orientation on V with basis $v_i = \text{vector from } a_0 \text{ to } a_i$



Defn 2: An orientation of V is a choice of one of the two generators of $H^n(V, V - \{0\}; \mathbb{Z})$

RMK Defn 1 is equivalent to Defn 2.

Given an orientation on V choose $\Delta^n \hookrightarrow V$ orientation preserving embedding s.t. the centroid of Δ^n is mapped to 0

Then, Δ^n generates $H_n(V, V - \{0\}; \mathbb{Z})$ and we can choose the dual generator of $H^n(V, V - \{0\}; \mathbb{Z})$
 $\cong \text{Hom}(H_n(V, V - \{0\}; \mathbb{Z}), \mathbb{Z})$

For vector bundles: Let V be a real topological \mathbb{R}^n bundle $V \xrightarrow{\pi} B$

Defn 1: An orientation of V is a choice of orientation of $V_b \forall b \in B$ s.t. near each point $b \in B$, \exists a local trivialization $\pi^{-1}(U) \cong U \times \mathbb{R}^n$ that is orientation preserving on each fiber, where \mathbb{R}^n has the standard orientation.

Defn 2 An orientation on V is a choice of one of the two generators of $H^n(V_b, V_b - \{v\}; \mathbb{Z}) \quad \forall b \in B$ s.t. $\forall b \in B$ there exist a neighbourhood U of $b \in B$ which restricts to the chosen generator at all points of U . $B \xrightarrow{\pi} V$ is the zero section

$$\begin{aligned} X_+ &= X \sqcup \{\ast\} \\ \Rightarrow X_+ \wedge Y &= \frac{(X \sqcup \{\ast\}) \times Y}{X_+ \times \{y_0\} \cup \{\ast\} \times Y} \\ &= \frac{X \times Y}{X \times \{y_0\} \cup \{\ast\} \times y_0} \\ &= \frac{X \times Y}{X \times \{\ast\}} \end{aligned}$$

Recall last time

$$Th(V) := V / V - \text{zero section}$$

$$V \xrightarrow{\pi} B$$

\square zero section

and that for X, Y pointed spaces the smash product

$$X \wedge Y = \frac{X \times Y}{X \times \{y_0\} \cup \{x_0\} \times Y}$$

$$\text{ex } S^m \wedge S^n = S^{m+n}$$

$$\begin{aligned} X_+ \wedge S^1 &= \frac{X \times S^1}{X \times \{1\}} \simeq \frac{X \times [0, 1]}{X \times \{0, 1\}} \\ &= \frac{X \times D^1}{X \times S^0} = \frac{D(\Theta_X)}{S(\Theta_X)} \end{aligned}$$

where $\Theta_X \rightarrow X$ is the trivial bundle of rank 1.

Q Why is $(X_+) \wedge S^n \simeq Th(\Theta_X^{\otimes n})$

Note that

$$\sum^n X = \underbrace{S^1 \wedge \dots \wedge S^1 \wedge X}_{n \text{ times}} = S^n \wedge X$$

$$\begin{aligned} X_+ \wedge S^n &= \frac{X \times S^n}{X \times \{1\}} = \frac{X \times D^n}{X \times S^{n-1}} \\ &\simeq \frac{D(\Theta_X^n)}{S(\Theta_X^n)} \end{aligned}$$

$\Theta_X^n \rightarrow X$ is the trivial rk n vector bundle

Defn 3: An orientation on V is an element

$$v \in H^n(Th(V); \mathbb{Z})$$

called a Thom class, s.t. the restriction $v|_{Th(V_b)}$ in $H^n(V_b, V_b - \{v\}; \mathbb{Z})$ is a generator $\forall b \in B$.

Defn 2 & 3 are equivalent
 $(3) \Rightarrow (2)$ is straightforward
 $(2) \Rightarrow (3)$:

Assume B is a CW cx, we may assume that B has finitely many cells.
 \Rightarrow by induction on the number of cells,

$$B = U_1 \cup U_2$$

where U_1 deformation retracts onto 1 fewer cell & U_2 deformation retracts onto

The Thom isomorphism states:

Let $V \xrightarrow{\pi} B$ be a rank n oriented vector bundle

$$\Rightarrow H^*(X; \mathbb{Z}) \rightarrow \tilde{H}^{*+n}(Th(V); \mathbb{Z})$$

$$\gamma \mapsto \gamma \cup \alpha$$

$$Ex: \mathbb{R}^n \xrightarrow{\pi} \{\ast\}$$

$$H^*(\{\ast\}; \mathbb{Z}) \cong H^{*+n}(S^n)$$

the last cell.

By Mayer-Vietoris

$$H^n(Th(V)) \rightarrow H^n(Th(\pi|_{U_1})) \oplus H^n(Th(\pi|_{U_2}))$$

$$\xrightarrow{r} H^n(Th(\pi|_{U_1 \cup U_2}))$$

By construction

$$\delta'_{U_1} \oplus \delta'_{U_2} \in \text{Ker}(r)$$

$\alpha \in H^n(Th(V); \mathbb{Z})$ as in defn 3 is called a Thom class
 $(\alpha \in \tilde{H}^n(Th(V); \mathbb{Z}))$

Proof of the Thom isomorphism thm.

Assume B is a CW complex

Case 1: B has finitely many cells.

We will induct on the # of cells

$$B = U_1 \cup U_2 \text{ s.t.}$$

where U_1 deformation retracts onto 1 fewer cell & U_2 deformation retracts onto the last cell

Then, where restriction maps commute with cup products

$$H^i(B) \rightarrow H^i(U_1) \longrightarrow H^i(U_2) \rightarrow H^i(U_1 \cap U_2)$$

$$\downarrow \cup \alpha \qquad \downarrow \cup \alpha \qquad \downarrow \cup \alpha$$

$$H^{n+i}(Th(V)) \rightarrow H^{n+i}(Th(\pi|_{U_1})) \rightarrow H^{n+i}(Th(\pi|_{U_1 \cap U_2})) \rightarrow H^{n+i}(Th(\pi|_{U_1 \cap U_2}))$$

Note that

$U \cap V_2$ deformation retracts onto the boundary of one cell.

$$\Rightarrow U \cap V_2 \xrightarrow{\text{def}} S^k$$

By induction

$$H^i(V_j) \simeq H^{n+i}(Tn(\pi|_{V_j}))$$
$$j=1, 2$$

Then,

$$\text{since } U \cap V_2 \sim S^k$$

$$Tn(\pi|_{U \cap V_2}) = Tn(\Theta_{S^k}^n)$$
$$\simeq (S^k_+)^n S^n$$

Then,

$$H^i(S^k) \longrightarrow H^{i+n}(Tn(\pi|_{U \cap V_2}))$$
$$\delta \mapsto \delta \cup u$$

$$\text{and } \partial(\delta \cup u) = (\partial\delta) \cup u + \delta \cup \partial u = 0$$

This map is clearly an isomorphism for $k > 2$. For $k \leq 2$ use:

Lemma $\tilde{H}^i(X) \simeq H^{i+1}(\Sigma X)$
 $\simeq H^{i+1}(S^1 \wedge X)$

Reality check:

$$X = * \vee * = S^0$$

$$X \wedge S^1 = S^1$$

$$\tilde{H}(S^0) \simeq \tilde{H}(S^1) \simeq \mathbb{Z}.$$

Therefore

$$H^i(B) \simeq H^{n+i}(Tn(V))$$

Case 2: B has infinitely many cells. Then we can write

$$B = \varinjlim B_i$$

B_i is a finite CW complex

$$\Rightarrow H^*(B) \xrightarrow{\cong} \varprojlim H^*(B_i)$$

and,

$$H^*(B) \xrightarrow{\cong} \varprojlim H^*(B_i) \downarrow \cup u \quad \begin{matrix} \text{this is an iso} \\ \text{by case 1} \end{matrix}$$
$$H^{k+n}(Tn(V)) \xrightarrow{\cong} \varprojlim H^*(Tn(\pi|_{B_i}))$$

Defn 4 of orientation
 An orientation of a rank n \mathbb{R} -topological vector bundle is a section σ of the \mathbb{Z} local system

$\mathcal{O}(V) \rightarrow B$
 the "orientation sheaf of V "
 with fiber
 $\mathcal{O}(V)_b = H^n(V_b, V_b - 0; \mathbb{Z})$
 s.t. $\sigma(b)$ is a generator V_b

Defn: For an oriented \mathbb{R}^n vector bundle V , the Euler class $e(V)$ is the pullback of the Thom class:

$e(V) = z^*(u)$
 where $z: B \xrightarrow{\sim} V$ is the zero section.

$$Th(V) = V / V - z(B)$$

$$H^*(Th(V)) \xrightarrow{z^*} H^*(B)$$

Rmk A C^n bundle determines a canonical oriented \mathbb{R}^{2n} bundle

$\Rightarrow e$ is a characteristic class

PROP If $V \rightarrow B$ is a

C^n bundle

$$e(V) = z^*(u) = c_n(V)$$