

Lecture 12

THM (cohomology of Grassmannians)

The map
 $\mathbb{Z}[c_1, \dots, c_n] \rightarrow H^*(\mathbb{C}G_{r,n}; \mathbb{Z})$

$$c_i \mapsto c_i(S)$$

is an isomorphism, and

$\mathbb{Z}_2[w_1, \dots, w_n] \rightarrow H^*(\mathbb{R}G_{r,n}; \mathbb{Z}_2)$

$$w_i \mapsto w_i(S)$$

is an isomorphism.

$S \rightarrow \mathbb{K}G_{r,n}$ is the tautological bundle (for $\mathbb{K} = \mathbb{C}$ or \mathbb{R})

Proof: (for \mathbb{C})

Last time we showed that the map

$$\underbrace{\mathbb{P}^\infty \times \dots \times \mathbb{P}^\infty}_{n \text{ times}} \rightarrow G_{r,n}$$

classifying $\bigoplus_i \pi_i^* \mathcal{O}(-1)$

induces a surjection

$$H^*(\mathbb{C}G_{r,n}; \mathbb{Z}) \rightarrow \mathbb{Z}[c_1, \dots, c_n]$$

$$c_i(S) \mapsto c_i \quad (*)$$

We NTS injectivity

Ehresmann "schubert cell"
 CW structure has $p(r,n)$
 r -dimensional cells:

\Rightarrow by CW homology

$$\dots \rightarrow \text{Hom}(\bigoplus_{r-1} \mathbb{Z}, \mathbb{Z}) \rightarrow \text{Hom}(\bigoplus_r \mathbb{Z}, \mathbb{Z}) \rightarrow \text{Hom}(\bigoplus_{r+1} \mathbb{Z}, \mathbb{Z}) \rightarrow \dots$$

$H^r(\mathbb{C}G_{r,n}; \mathbb{Z})$ is gen'd by $\leq p(r,n)$ generators.

By classification of abelian groups we have that

$$H^a(\mathbb{C}G_{r,n}; \mathbb{Z}) = \mathbb{Z}^a \oplus \mathbb{Z}/d_1 \oplus \dots \oplus \mathbb{Z}/d_l$$

with $d_1 | d_2 | \dots | d_l$

and

$$\text{deg}_r(\mathbb{Z}[c_1, \dots, c_n]) \cong \mathbb{Z}^{p(r,n)}$$

By (*) we have a surjection

$$\mathbb{Z}^a \oplus \mathbb{Z}/d_1 \oplus \dots \oplus \mathbb{Z}/d_l \twoheadrightarrow \mathbb{Z}^{p(r,n)}$$

Tensoring with \mathbb{Q}
 gives

$$\mathbb{Q}^a \twoheadrightarrow \mathbb{Q}^{p(r,n)}$$

$$\Rightarrow a \geq p(r,n)$$

$$\Rightarrow a = p(r,n)$$

$$\Rightarrow l = 0.$$

Thus,

$$\begin{aligned}
 H^v(\mathbb{C}Gr_n; \mathbb{Z}) &\cong \mathbb{Z}^{p(r,n)} \\
 &\downarrow \\
 &\text{degr}(\mathbb{Z}[c_1, \dots, c_n]) \\
 &\cong \mathbb{Z}^{p(r,n)}
 \end{aligned}$$

It remains to be shown that this map is injective. In fact any surjective map $\mathbb{Z}^{p(r,n)} \rightarrow \mathbb{Z}^{p(r,n)}$ is an isomorphism

Let $K = \text{Kernel}$, $\bar{p} = p(r,n)$

$$0 \rightarrow K \rightarrow \mathbb{Z}^{\bar{p}} \rightarrow \mathbb{Z}^{\bar{p}} \rightarrow 0$$

K is a submodule of a f.g free module over a PID.

$\Rightarrow K$ is free & f.g
 $K \cong \mathbb{Z}^m$

\Rightarrow tensor the SES above with \mathbb{Q} get SES of vector spaces

$$0 \rightarrow \mathbb{Z}^m \otimes \mathbb{Q} \rightarrow \mathbb{Z}^{\bar{p}} \otimes \mathbb{Q} \rightarrow \mathbb{Z}^{\bar{p}} \otimes \mathbb{Q} \rightarrow 0$$

$\Rightarrow m = 0$ □

Q What is $c(\det S^* \rightarrow Gr_n)$?

Note that $(\det S^*) = (\det S)^*$ since the dual commutes with determinant b/c Hom & wedge commute.

$$\det S^* \otimes \det S = \mathcal{O} \text{ trivial line bundle}$$

We also know $c(\det V) = 1 + c_1(V)$

Since it's a line bundle

$$\begin{aligned}
 c(\det S^*) &= 1 + c_1(\det S^*) \\
 &= 1 + c_1((\det S)^*) \\
 &= 1 - c_1(\det S) \\
 &= 1 - c_1(S)
 \end{aligned}$$

$$\begin{aligned}
 H^*(\mathbb{C}Gr_n; \mathbb{Z}) &\cong \mathbb{Z}[c_1, \dots, c_n] \\
 c_i(S) &\mapsto c_i
 \end{aligned}$$

$$\Rightarrow c(\det S^*) = 1 - c_1$$

In general we can ask if $\text{Hom}(V, V) \cong V^* \otimes V$ is trivial

e.g $V = \mathcal{O}(1) \oplus \mathcal{O}(2) \rightarrow \mathbb{P}^1$

$$\Rightarrow V^* = \mathcal{O}(-1) \oplus \mathcal{O}(-2) \rightarrow \mathbb{P}^1$$

$$\begin{aligned}
 V \otimes V^* &= (\mathcal{O}(1) \oplus \mathcal{O}(2)) \otimes (\mathcal{O}(-1) \oplus \mathcal{O}(-2)) \\
 &= \mathcal{O}(1) \oplus \mathcal{O}(-1) \oplus \mathcal{O} \oplus \mathcal{O}
 \end{aligned}$$

So for this V the answer is No.

Another ex:
 $V = TM$
 $V^* = T^*M \cong TM$
& $TM \otimes TM$ is not often trivial.

Suspensions

For pointed spaces
 $x_0 \in X, y_0 \in Y$

$$X \wedge Y = X \times Y / x_0 \times Y \cup X \times y_0$$

$$\text{Ex } (S^1, 1) \wedge (S^1, 1) = S^2$$

$$\Sigma X = S^1 \wedge X$$

Note that

$$\Sigma S^2 = S^1 \wedge S^2 = S^3$$

$$\& \Sigma S^n = \Sigma^n S^1$$

Suppose we have a CW X
 with cells:

0-cells 1 cells 2 cells



\Rightarrow cells of $\Sigma^n X$ are

0 cells ... n cells n+1 cells n+2 cells



From CW homology

$$\tilde{H}^k(X; \mathbb{Z}) = \text{Ker}(H^k(X; \mathbb{Z}) \rightarrow H^k(X_0; \mathbb{Z}))$$

$$\tilde{H}^k(\Sigma^n X) \cong H^{k-n}(X)$$

Thom spaces: Given a $V \rightarrow X$
 vector bundle with a Euclidean metric
 $DV = \{v \mid \|v\| \leq 1\}$

$$SV = \{v \mid \|v\| = 1\}$$

Then the Thom space of V is
 $\text{Th}(V) = DV/SV$

Q Let $V \rightarrow X$ be
 the trivial bundle
 of rank n

$V = \mathbb{O}^n$. What is
 $\text{Th}(V)$ in terms of
 suspensions?

Note that

$$\text{Th}(V) = D^n \times X / S^{n-1} \times X$$

$$X_+ = X \sqcup \{*\}$$

$$\begin{aligned} \Sigma^n X_+ &= S^n \wedge X_+ = X_+ \wedge (D^n / S^{n-1}) \\ &= X_+ \times (D^n / S^{n-1}) \\ &= \{*\} \times D^n / S^n \cup X_+ \times \{*\} \end{aligned}$$

$$\begin{aligned} &\stackrel{\text{equiv}}{=} X \times D^n / (X \times S^{n-1}) \sqcup (\xi * \zeta \times D^n) \\ &= X \times D^n / X \times S^{n-1} \end{aligned}$$

Fact: $Tn(\mathbb{O}^n) = \Sigma^n X_+$

& the cells of $\Sigma^n X_+$ are the i -cells of X and one extra 0 cell

Let $V \rightarrow X$ be a rank n \mathbb{R} vector bundle

Let X be a CW c.x w/ cell structure s.t. V is trivial over each cell

\Rightarrow i -cells of $Tn(V)$ are the i -cells of X and one extra 0-cell.

$$H^*(Tn(V)) = H^*(D^n \times X / S^{n-1} \times X)$$

Then isomorphism thm

$$\tilde{H}^*(Tn(V)) \cong H^{*-n}(X)$$

$$H^*(DV, SV; \mathbb{Z})$$

hypothesis on orientation necessary.

Other ways to write $Tn(V)$

$$Tn(V) = \frac{D(V)}{S(V)} \stackrel{\text{hntpy equiv}}{\cong} \frac{D(V)}{D(V) - Z}$$

$Z: X \rightarrow V$ is the zero section

$$\frac{D(V)}{D(V) - Z} \stackrel{\text{hntpy equiv}}{\cong} \frac{V}{V - Z}$$

and if we don't want to pick a metric

$$P(V \oplus \mathbb{O}) = \underset{\substack{\uparrow \\ \text{cod } X_0}}{V} \underset{X_0 \neq 0}{\sqcup} P(V) \underset{X_0 = 0}{\sqcup} P(V)$$

By excision

$$\begin{aligned} &P(V \oplus \mathbb{O}) / P(V \oplus \mathbb{O}) - Z(X) \\ &\cong V / V - Z(X) \cong Tn(V) \end{aligned}$$