

## Lecture 10

Let  $E = \mathcal{O}(1) \oplus \mathcal{O}(2) \xrightarrow{\pi} \mathbb{P}^3$   
 be the Whitney sum of the dual  
 tautological bundle & its second  
 tensor power.  
 What is  $H^*(\mathbb{P}E)$ ? (as a ring)

First note that the fiber  
 $\pi^{-1}(b)$  has homology  
 $\mathbb{R}[t]/t^2$  and that one  
 can lift  $t$  to  $H^2(\mathbb{P}E)$   
 using the tautological  
 bundle on  $\mathbb{P}^\infty$

we can therefore use Leray  
 Hirsch.

$$\Rightarrow H^*(\mathbb{P}E) \cong H^*(\mathbb{P}^3) \otimes H^*(\mathbb{P}E|_b)$$

as  $\mathbb{R}$ -modules

$$\cong \mathbb{Z}[t, s] / \langle s^4, (t+s)(t+2s) \rangle$$

$$\begin{aligned} \text{since } 0 &= t^2 + c_1(E)t + c_0(E) \\ &= (t + c_1(\mathcal{O}(1))) (t + c_1(\mathcal{O}(2))) \\ &= (t+s)(t+2s) \end{aligned}$$

Principle [Grothendieck] A projective bundle formula gives Chern  
 classes & vice-versa.

THM [Splitting Principle] For a rank  $n$  bundle  $V \rightarrow B$  there is a  
 map  $f: B' \rightarrow B$  s.t.  $f^* V \cong L_1 \oplus \dots \oplus L_n$  where  $L_i \rightarrow B'$   
 is a rank 1 bundle for  $i=1, \dots, n$   
 $f^*: H^*(B; \mathbb{R}) \rightarrow H^*(B'; \mathbb{R})$  is injective  
 (we assume  $B$  is a topological paracompact & Hausdorff space)  
 (if  $V$  is a  $\mathbb{C}^n$  vector bundle  $R = \mathbb{Z}_2$   
 & if  $V$  is an  $\mathbb{R}^n$  vector bundle  $R = \mathbb{Z}$ )

Proof: by induction. It's true for  $n=1$ .

Suppose  $V \rightarrow B$  is a rank  $n$  vector bundle. There is a tautological  
 injection

$$f_1: \mathbb{P}V \rightarrow B \Rightarrow \mathcal{O}(-1)_{\mathbb{P}V} \hookrightarrow f_1^* V$$

$$\mathcal{O}(-1)_{\mathbb{P}V} \xrightarrow{\pi} \mathbb{P}V$$

where  $\pi^{-1}([\ell]) = \{v \in V_b \mid v \in \ell\}$   
 and  $\ell$  is a line in  $V_b$

$$\begin{array}{ccc} \Theta_{\mathbb{P}V}(-1)_{[l]} & \xrightarrow{\quad f^* V \quad} & V_b/l \\ \downarrow v \quad \quad \quad \parallel & & \\ V \in & V_f[l] & \\ & \parallel & \\ & V_b & \end{array}$$

$$\text{let } V' = f^* V / \Theta_{\mathbb{P}V}(-1)$$

$$0 \rightarrow \Theta_{\mathbb{P}V}(-1) \rightarrow f^* V \rightarrow V' \rightarrow 0$$

$$\Rightarrow f^* V \cong V' \oplus \Theta_{\mathbb{P}V}(-1)$$

By induction we have  $f_2 : B' \rightarrow \mathbb{P}V$   
satisfying the hypothesis

$$\Rightarrow f = f_2 \circ f_1 \\ \text{& } f^* V \cong L_1 \oplus \dots \oplus L_n$$

By Leray Hirsch

$$f^* : H^*(B) \rightarrow H^*(\mathbb{P}V) \text{ injective}$$

$$\text{by induction } f_2^* \text{ injective} \Rightarrow f^* \text{ injective}$$

□

$$\text{Then } c_1(L) = f^* t = f^*(\Theta(1))$$

(2) & the splitting determine all the chern classes.

THM [Uniqueness of SW & Chern classes]

There is at most one sequence of functors

$$\begin{aligned} c_i : \text{Vect}^n(B) &\rightarrow H^{2i}(B; \mathbb{Z}) \text{ s.t.} \\ (1) \quad f^* c_i(V) &= c_i(f^* V) \\ (2) \quad c_i(V_1 \otimes V_2) &= c_i(V_1) \cup c_i(V_2) \\ c_i \text{ total chern classes} \\ (3) \quad c_i(V) &= 1 \quad c_i(V) = 0 \quad i > \text{rk}(V) \\ (4) \quad c_i(\Theta(1)) &= t \in H^2(\mathbb{P}^1) \end{aligned}$$

Proof

(4) implies that

$$c_i(\Theta(1) \rightarrow \mathbb{P}^\infty) = t \in H^2(\mathbb{P}^\infty)$$

$\parallel$

$\mathbb{Z}[t]$

because

$\mathbb{P}^1 \hookrightarrow \mathbb{P}^\infty$  induces an isomorphism on  $H^2$ .

For any line bundle  $L \rightarrow B$   
 $\exists f : B \rightarrow \mathbb{P}^\infty$  by  
classification of topological vector bundles.

SW & Chern classes of the tensor product.

PROPOSITION: Let  $L_i \rightarrow X$  be rank 1  $\mathbb{C}$ -vector bundles  
 $\Rightarrow c_i(L_1 \otimes L_2) = c_i(L_1) + c_i(L_2)$

$\exists$  an analogous statement for SW classes

Proof

CASE 1:  $X = X_1 \times X_2$  where  $X_1 = X_2 = \mathbb{P}^\infty$

Then we have projection maps

$$\begin{array}{ccc} X_1 \times X_2 & \xrightarrow{\pi_1} & X_1 \\ & \searrow & \\ & X_2 & \end{array}$$

Let  $L_i = \pi_{i*}(\mathcal{O}(-1))$

where  $\mathcal{O}(-1) \rightarrow \mathbb{P}^\infty$

$$H^*(X_1 \times X_2) = \mathbb{Z}[t_1, t_2]$$

Künneth

where  $t_1, t_2$  are in degree 2

We can map

$$X_1 \vee X_2 \xrightarrow{i} X_1 \times X_2$$

which is an isomorphism on

$H^2$ ,

$$\begin{aligned} H^2(X_1 \vee X_2) &\cong H^2(X_1 \times X_2) \\ &\cong H^2(X_1) \oplus H^2(X_2) \\ &\cong \mathbb{Z}t_1 \oplus \mathbb{Z}t_2 \end{aligned}$$

$$\Rightarrow i^* c_1(L_1 \otimes L_2) = at_1 + bt_2$$

for integers  $a, b \in \mathbb{Z}$

$$i^* c_1(L_1 \otimes L_2) = c_1(i^* L_1 \otimes i^* L_2)$$

$$j_i^* L_i = \mathcal{O}(-1) \quad \text{where}$$

$$X_i \hookrightarrow X_1 \vee X_2 \xrightarrow{i} X_1 \times X_2$$

$j_i$

$i^* L_1 \otimes i^* L_2$  is a bundle whose restriction to  $X_i$  is  $\mathcal{O}_{X_i}(-1)$ , by naturality  $a = b = -1$

□

$$\Rightarrow c_1(L_1 \otimes L_2) = c_1(L_1) + c_1(L_2)$$

The general case will be covered in the next lecture.