

# Lecture 1

A **vector bundle** is  $(E, B, p)$

$E$   $E$  is the total space  
 $\downarrow p$   $B$  is the base space &  
 $B$   $p^{-1}(b)$  has the structure  
of a vector space  $\forall b \in B$  &  
 $p$  is locally trivial

i.e.  $\exists$  cover  $B = \bigcup_{\alpha} U_{\alpha}$  open  
sets such that

$$p^{-1}(U_{\alpha}) \cong_{i_{\alpha}} U_{\alpha} \times V_{\alpha}$$



$i_{\alpha}$  respects vector space  
structure

**RMK** This definition makes  
sense if  $(E, B, p)$  are

- (1) Topological spaces
- (2) Smooth manifolds
- (3) Complex analytic spaces
- (4) Schemes

In each case  $i_{\alpha}$  has to respect the  
relevant structure

- (1) homeomorphism
- (2) smooth homeo with smooth  
inverse
- (3) isomorphism of  $\mathbb{C}$ -analytic  
spaces  
(holomorphic with holomorphic)  
inverse
- (4) regular map with regular  
inverse, isomorphism of  
schemes.

## EXAMPLES

(of topological) vector  
bundles)

$S^1 \times \mathbb{R}$



Möbius band



$$\mathbb{R}P^1 = \mathbb{R}^2 - 0 / (x, y) \sim (\lambda x, \lambda y)$$

for  $\lambda \in \mathbb{R}^+$

= space of lines through  
origin in  $\mathbb{R}^2$

$$\mathbb{R}P^1 \cong S^1$$

**Defn:** We will call  
 $S \xrightarrow{p} \mathbb{R}P^n$

$$p^{-1}([L]) = L$$

the **tautological bundle**

(can replace "R" by "C")

Note that in Milnor Stasheff they call the tautological bundle the canonical bundle. We don't because the canonical bundle is something else in algebraic geometry

Algebraic notation  
 $\mathcal{O}(-1) \rightarrow \mathbb{P}^n$   
 tautological bundle

**EXAMPLES** of vector bundles

• If  $X$  is a smooth manifold or a scheme

$TX \rightarrow X$  tangent bundle

•  $Z \hookrightarrow X$  embedding of manifolds

$N_Z X \rightarrow Z$  normal bundle

$$(N_Z X)_{z_0} = T_{z_0} X / T_{z_0} Z$$

$z_0 \in Z$

Q What are characteristic classes? A functor

vector bundles on  $B \rightarrow$  cohomology of  $B$

they can distinguish two smooth structures on  $M$  if characteristic classes on  $TM$  are different.

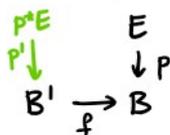
THM [Milnor]  $\exists$  exotic 7-spheres

Defn Given a vector bundle

$E \xrightarrow{p} B$  and a map  $f: B' \rightarrow B$

the **pullback**  $p^*E \xrightarrow{p'} B'$  is a vector bundle with fiber

$$(p^*E)_{b'} = E_{f(b')}$$



Defn A **characteristic class** is a functor

$$c_i: \{ \text{V-bundles on } B \} \rightarrow H^i(B; A)$$

for fixed  $i \in \mathbb{Z}$ , abelian group  $A$ , vector space  $V$ , varying  $B$

s.t.  $c_i(p^*E) = f^*(c_i(E))$

RMK Cohomology can be more general than  $H^i(-; A)$

Example of a characteristic class: Euler class  $e$

Given  $V = \mathbb{R}^n$ ,  $i = n$ ,  $A = \mathbb{Z}$   
 & restricting to oriented vector bundles  $(E, B, p)$ ,  
 $B$  oriented manifold of dim  $n$   
 $\Rightarrow H^n(B; \mathbb{Z}) = \mathbb{Z}$

$e(TB) = \chi(B)$  Euler characteristic  
 has many interpretations

$$\begin{aligned} e(TB) &= \chi(B) \\ &= \sum_i (-1)^i \# \{i \text{ cells}\} \\ &= \sum_i (-1)^i \text{rk}(H^i(B; \mathbb{Q})) \\ &= \frac{1}{\text{Vol } S^n} \int_B K \text{dvol} \\ &= \text{self intersections of } B \\ &\quad \text{inside } TB \text{ (after perturbation)} \\ &= \sum \text{ind}_p(S) \quad \text{where we} \\ &\quad \text{zeros } p \quad \text{choose any} \\ &\quad \text{s.t. } S(p) = 0 \quad \text{vector fields} \\ &\quad \quad \quad \quad \quad \text{w/ only} \\ &\quad \quad \quad \quad \quad \text{isolated zeros} \end{aligned}$$

### Grassmanians

$\mathbb{R} \text{Gr}(n, m) =$  moduli space of  
 $\mathbb{R}^n$ 's in  $\mathbb{R}^m$   
 through the  
 origin

$\mathbb{R} \text{Gr}(n) = \text{colim}_{m \rightarrow \infty} \mathbb{R} \text{Gr}(n, m)$   
 $=$  moduli space of  $\mathbb{R}^n$ 's  
 in  $\mathbb{R}^\infty$  through  
 the origin.

**RMK** If  $\exists$  nonvanishing  $S$   
 vector field on  $B$   
 $\Rightarrow e(TB) = 0 = \chi(B)$ .  
 Note that for  $S^2$   
 $\chi(S^2) \neq 0 \Rightarrow e(TS^2) \neq 0$   
 $\Rightarrow TS^2$  is not trivial.

We have a tautological bundle

$$S \xrightarrow{p} \text{Gr}(n, m)$$

$\cup$   
 $[P]$   
 plane

$p^{-1}([P]) = P$

Q How many lines meet 4 lines in  $\mathbb{P}^3$ ?

A  $e\left(\bigoplus_{i=1}^4 S^* \wedge S^*\right)$  where  $S \xrightarrow{P} \text{Gr}(2,4)$

An essential result for topological vector bundles:

THM we have a bijection  
 $\left\{ \begin{array}{l} \text{Topological } \mathbb{R}^n \\ \text{bundles on } B \end{array} \right\} / \text{iso} \longleftrightarrow \left\{ \begin{array}{l} \text{conts maps} \\ f: B \rightarrow \text{Gr}(n) \end{array} \right\} / \text{homotopy}$

$$V_f := f^* S \longleftarrow f$$

Thus

$$\begin{array}{ccc} E \cong f^* S & & S \\ \downarrow & \searrow P & \downarrow P \\ B & \xrightarrow{f} & \text{Gr}(n) \end{array}$$

Same for  $\mathbb{C}$  instead of  $\mathbb{R}$

The result is that:

characteristic classes are valued in cohomology of Grassmannians

$\left\{ \begin{array}{l} \text{characteristic classes} \\ \text{for topological } \mathbb{R}^n \text{ bundles} \\ \text{valued in } H^i(-, A) \end{array} \right\} \longleftrightarrow H^i(\mathbb{R} \text{Gr}(n); A)$

$$f^* c = c(V_f) \longleftarrow c$$