Some trig identities:

\[ \sin^2(x) + \cos^2(x) = 1 \]
\[ \tan^2(x) + 1 = \sec^2(x) \]
\[ 1 + \cot^2(x) = \csc^2(x) \]
\[ \sin(x + y) = \sin(x)\cos(y) + \cos(x)\sin(y) \]
\[ \cos(x + y) = \cos(x)\cos(y) - \sin(x)\sin(y) \]
\[ \cos^2(x) = \frac{1 + \cos(2x)}{2} \]
\[ \sin^2(x) = \frac{1 - \cos(2x)}{2} \]

And the hyperbolic trig identities:

\[ \cosh^2(x) - \sinh^2(x) = 1 \]
\[ 1 - \tanh^2(x) = \text{sech}^2(x) \]
\[ \coth^2(x) - 1 = \text{csch}^2(x) \]
\[ \sinh(x + y) = \sinh(x)\cosh(y) + \cosh(x)\sinh(y) \]
\[ \cosh(x + y) = \cosh(x)\cosh(y) + \sinh(x)\sinh(y) \]
\[ \cosh^2(x) = \frac{\cosh(2x) + 1}{2} \]
\[ \sinh^2(x) = \frac{\cosh(2x) - 1}{2} \]
1. Evaluate the sum of the following convergent series. Pay attention to where the series begin (5 points each):

(a) $\sum_{n=1}^{\infty} \frac{n}{3^n}$

(b) $\sum_{n=1}^{\infty} \left(e^{1/n} - e^{1/(n+1)}\right)$

(c) $1 - 1 + \frac{1}{2} - \frac{1}{3!} + \frac{1}{4!} + ...$

(d) $\sum_{n=1}^{\infty} \frac{3^n}{4^n}$
2. Determine whether the following series converge absolutely, converge conditionally, or diverge. State for each the test(s) used and show how the hypotheses of the test(s) are satisfied (5 points each):

(a) \[ \sum_{n=1}^{\infty} \frac{1}{n^{1/\ln(n)}} \]

(b) \[ \sum_{n=2}^{\infty} (-1)^n (\sqrt{n} - \sqrt{n-1}) \]
(c) $\sum_{n=1}^{\infty} n \sin\left(\frac{1}{n}\right)$

(d) $\sum_{n=1}^{\infty} \frac{n+2^n}{n+3^n}$
3. (a) (8 points) Find $P_3(x)$, the third-degree Taylor polynomial for $f(x) = \ln(x)$ at $a = 1$.

(b) (7 points) Find an upper bound for the error if the $P_3(x)$ found in part (a) is used to estimate $\ln(x)$ for $|x - 1| \leq 0.2$. 
4. (10 points) Use series to approximate \( \int_0^1 \frac{1 - \cos(x^2)}{2} dx \) with error < .0005. Write your answer like \( 1 - 1 + \frac{1}{2} - \frac{1}{6} + \frac{1}{24} \) (no need to do the arithmetic necessary to combine the terms). Clearly indicate the smallest number of terms needed and how you know that number is enough.

5. (5 points) What is the recurrence relation if we use our power series technique to start solving the differential equation \( y' + 2xy = 0 \)? Do NOT solve the equation all the way. I just want to see the recurrence relation.
6. (15 points) Determine where the power series \( \sum_{n=2}^{\infty} \frac{(x-1)^n}{3^n n^{3/2}} \) converges absolutely, converges conditionally, and diverges. Name all the tests you use and show that the hypotheses of the tests are satisfied.

converges absolutely  
converges conditionally 
diverges
7. (15 points) Use power series to find the solution of the differential equation \( y'' - 2y' + y = 0; \ y(0) = 0, y'(0) = 1 \). Give the radius of convergence of the resulting series, and identify the general solution in terms of familiar elementary functions.
Please read the following and sign below.

I have upheld the Duke Community Standard in completing this examination.

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