Math 563, Spring 2020
Last updated: 4/13/20

HOMEWORK 9 (DUE WED. APR. 22)

Reading (suggested): Chapter 9 of Leveque.

Code to turn in: Your code for Problem 1 (optional: also the Crank-Nicolson code).

Note: Consider the problems not marked [AC] as the essential problems, and the [AC] problems as extensions, further examples and theoretical details.

1. Problems

Problem 1 (A typical non-linear BVP). Consider a steady state problem for the **porous medium equation** given by

$$(h^3 h')' = f(h), \quad x \in [0,2], \quad h(0) = h_\ell, \quad h(2) = h_r$$

which describes the height of a puddle of liquid with an external force $f(x)$.[1]

a) Write the ODE in the form $(g(h))'' = \cdots$. Then derive the Newton iteration required to solve the BVP using centered differences (don't convert to a first order system; while you can solve this as a linear BVP for $g(h)$, keep it as a system for h itself).

b) Implement your scheme. While it is possible to simplify, you should write the boundary-affected parts outside the main loop. Obtain a solution when $f(x) = x^2$, $h_\ell = 1$ and $h_r = 2$ with a maximum error of 10^{-6} and provide evidence that your method is convergent with the appropriate order. [Thorough version: pretend you don't know the exact solution. Quick version: use the exact solution to find the error].

c) [AC] Let $q = h^3 h'$ and derive, in detail, the Newton iteration required to solve the BVP using the midpoint scheme for the first order system involving (h, q).

Problem 2 (Stability, adapted from A&P 8.11) [AC]. Consider the linear boundary value problem

$$-y'' + ay' = q(x), \quad y(0) = c, \quad y(1) = d$$

Two 'nice' properties that are desirable for numerical stability are:

- A is **diagonally dominant**: the absolute sum of the non-diagonal entries in each row are at most the size of the diagonal entry ($\sum_{j=1, j\neq i}^{n} |a_{ij}| \leq a_{ii}$ for each i)
- The sign pattern in each row is $-, +, -$ (e.g. $-1, 2, -1$) with $+$ on the diagonal (or the opposite).

(This, for instance, ensures A is positive definite, and that LU decomposition can be done stably without pivoting).

[1]The 'porous medium equation' is the PDE $u_t = (u^n u_x)_x$ for an integer n, which is an important nonlinear diffusion equation.

a) Write the problem to be solved for the approximation as $Au = b$. Use centered differences and a uniform grid with $u_0 = y(0)$ and $u_{N+1} = y(1)$ and spacing h.

b) Show that the matrix in (a) is only has these properties if $R = |a|h < 2$ (this value is called the 'grid Reynolds number').

c) Assuming $a > 0$, show that if the first order derivative is instead discretized using a backward difference, then there is no restriction on R to have the nice properties (this is a common place where upwind discretization is used.

Problem 3 (a typical heat equation problem). The **Crank-Nicolson method** is a popular second-order method (in both time and space) for solving heat-like equations.

a) Consider the heat equation $u_t = au_{xx}$. Use the method of lines, with centered differences in space and the **implicit trapezoidal method** in time to derive the Crank-Nicolson method in the form

$$\frac{U_n^{k+1} - U_n^k}{\Delta t} = \cdots$$

b) Derive an explicit expression for the truncation error (up to small higher-order terms) involving only x derivatives in u.

c) Use Von Neumann analysis to determine the stability restriction.

d) Implement the method and use it to solve

$$u_t = au_{xx}, \quad x \in [0, \pi]$$

with $a = \frac{1}{4}$, boundary conditions

$$\frac{\partial u}{\partial x}(0, t) = -\frac{1}{t+1}, \qquad \frac{\partial u}{\partial x}(\pi, t) = 0$$

and initial condition

$$u(x, 0) = \sin x.$$

in the time interval $[0, 4]$. Show (with a convergence plot) that the method is indeed second order in Δt and Δx. To do so, consider the error

$$E(T) = \max_{0 \le n \le N} |U_n^K - u(x_n, T)|$$

where $T = 4$ is the final time with at least one of two approaches:

(i) Use a small Δx and Δt to get an approximate 'exact' solution, then plot E(T) etc.[2]

(ii) Estimate the convergence order p using the table of p's (not requiring an exact solution) as done before. (You may need to sample a point instead of using the max. error over x).

---

[2]You could compute the exact solution analytically, but pretend that is not possible here. The Aitken extrapolation tricks aren't really convenient here, so the crude approach is easier.

Problem 4 (More on upwind). Consider the advection equation

$$u_t + cu_x = 0$$

and the method using forward differences for time and **backward** differences in space.

a) Use Von Neumann analysis to show that there is a CFL condition on $\Delta t/\Delta x$ **and** that the speed c must have a certain sign. (What should be done to modify the method if $c < 0$? What if $c = c(x)$ depends on x?).

b) Find the modified equation and show that you get the same stability restriction as in (a). [AC] Compare this to the modified equation for Lax-Friedrichs: which has more diffusion?

c) [AC, a numerical example] Consider the problem

$$u_t + (u^2)_x = 0, \quad u(0,t) = 1$$

with the initial condition $u(x,0) = e^{-x^2}$. Solve this numerically using the method in (a). What happens to the solution, and why does this indicate some care must be taken in solving such problems numerically?

Problem 5 (diffusing a grid) [AC]. Suppose we have a set of points $\{x_j\}$ in $[0,L]$ with $0 = x_0 < \cdots < x_{N+1} = L$ and wish to 'smooth them out' so they are a bit more evenly distributed (but still retain some of its original configuration). To be precise, we want:

- The $\tilde{x}_j$'s are still close to the distribution of the x_j's (as much as possible) but
- The ratio of successive Δx's stays between factors $1/\delta$ and δ; that is if $\Delta\tilde{x}_j = \tilde{x}_{j+1} - \tilde{x}_j$ then
$$\frac{1}{\delta} \le \frac{\Delta\tilde{x}_j}{\Delta\tilde{x}_{j-1}} \le \delta.$$

a) Construct a grid of points in $[0,2]$ with two values of Δx: a 'high-resolution' region $[0.9, 1.1]$ with a spacing of $\Delta x = 10^{-3}$ an a 'low-resolution region' everywhere else with $\Delta x = 10^{-2}$. What is the resulting value of N?

a) View x as a function $\chi(q)$ where the q's are evenly distributed in $[0,1]$ (so $q_j = j/(N+1)$ for $j = 0, \cdots, N+1$ and $x_j = \chi(q_j)$). Write a scheme for solving the heat equation

$$x_t = x_{qq}, \quad q \in [0,1], \qquad x(q=0,t) = \chi(q)$$

with the appropriate boundary conditions. This allows you to 'diffuse' the points by running the heat equation. What would happen if you ran your solver for a long time?

b) Implement this scheme and use it to smooth out the given distribution to achieve the desired spacing given a value of δ (pick e.g. $\delta = 4$ as an example).

c) Could you 'sharpen' the distribution of points by running the solver in reverse?