Math 563, Spring 2020
Last updated: 4/5/20

HOMEWORK 8 (DUE FRIDAY. APR. 8)

Reading (suggested): Chapter 7 of Ascher & Petzold is a good overview of shooting. For an introduction to finite differences, see Chapter 2 (esp. 2.1-2.2, 2.14 and 2.16) of R.J. Leveque's *Finite Difference Methods for Ordinary and Partial Differential Equations* (this is a good book!).

Code to turn in: Your code for Problems 2 and 3.

1. Problems

Problem 1 (a typical shooting problem).
In fluid dynamics, the **Blasius boundary layer equation**[1] is

$$f''' + \frac{1}{2}f''f = 0, \quad f(0) = f'(0) = 0, \quad \lim_{x\to\infty} f'(x) = 1.$$

Use shooting to solve for $f(x)$, replacing the boundary condition at ∞ with $f'(L) = 1$. Pick a value of L that is 'large enough'. Obtain a value of $f''(0)$ that is accurate to four significant digits. For [AC], also justify your answer and choice of L numerically.

Problem 2 (Eigenvalues). Consider the (linear) eigenvalue problem

$$u'' = -\lambda x u, \quad x \in [0,4], \qquad u(0) = 0, \quad u(4) = 0$$

along with the normalization condition

$$\nu(4) = 1 \text{ where } \nu(x) = \int_0^x (u(x))^2\, dx.$$

a) State the IVP for shooting as a first-order system of three ODEs for $\mathbf{y} = (y_1, y_2, y_3)$, with shooting parameters $s_1 = y'(0)$ and $s_2 = \lambda$ (note: I am not counting $\lambda' = 0$ as one of the ODEs; treat λ as a parameter).

b) Derive the IVPs to solve for the variations $\mathbf{v} = \partial\mathbf{y}/\partial s_1$ and $\mathbf{w} = \partial\mathbf{y}/\partial s_2$. State the appropriate goal function and a formula for its Jacobian that can be computed.

c) Implement shooting as set up in (a) and (b) and use it to find the first five eigenvalues. Make a table of the computed values (computed to a reasonable accuracy like five significant digits) and a plot showing the first three eigenfunctions (one plot with all three). *Hint:* $\lambda_1 \approx 0.3$ *and the first five* λ*'s are all less than 10.*

d) [AC] Another strategy for 'computing' the Jacobian of the goal function $G(\vec{s})$ is to estimate the partials $\partial G_i/\partial s_j$ using finite differences (e.g. a forward difference), which is close enough for Newton to still work. How does the efficiency (ignoring accuracy/stability) compare to the method in (b)? Is there any advantage to doing this instead of (b)? *Hint: consider what IVPs need to be solved.*

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[1] The function $f'(x)$ describes the velocity profile of a fluid flowing past a flat surface with a velocity of 1 far away; the boundary layer arises because the fluid velocity must be zero at the surface (leading to drag).

Problem 3 (Continuation). (corrected) **Bratu's problem** in one dimension is

$$y'' = -\lambda e^y, \quad x \in [0,1], \qquad y(0) = y(1) = 0$$

which is a non-linear eigenvalue problem. It is known that there is a continuous family of solutions and there is a critical value λ^* such that

- There are two solutions for $0 < \lambda < \lambda^*$
- There are no solutions for $\lambda > \lambda^*$

While λ could be used as a continuation parameter, there is a problem due to the properties above.

a) Consider adding the norm condition

$$\int_0^1 (y(x))^2 \, dx = A^2.$$

A solution exists for each value of $A > 0$. Use the same trick as in P2 to convert the problem into one for $\mathbf{y} = (y_1, y_2, y_3)$ that can be solved using shooting with $y'(0)$ and λ as shooting parameters. (The code should be similar to P2).

Then solve the BVP for A in the range $[0, 4]$ and make a plot of the eigenvalue λ vs. A. Use continuation to get the right guesses for shooting. Use this data to estimate the value of λ^*.

Problem 4 (discretization vs. linearization). Consider the BVP

$$y'' = f(y), \quad y'(0) = c, \quad y'(b) = 0$$

where $f(y)$ is smooth. There are two basic approaches to a finite difference method:

a) ('Discretize, then linearize') Discretize the system with a uniform grid and centered differences and write out the non-linear system to be solved. Then derive the linear system to be solved in each step of Newton's method.

b) ('Linearize, then discretize') Suppose we have a function $y_k(x)$ that is close to the solution. We can try to adjust it with a small correction to get the solution to the BVP. That is, we seek a function $\eta(x)$ such that

$$y_{k+1}(x) = y_k(x) + \eta(x)$$

solves the BVP.

(i) Plug this expression into the BVP and linearize, discarding $O(\eta^2)$ terms, to obtain a **linear** BVP that can be solved for the correction η.

(ii) Use the same discretization scheme as in (a) to obtain a linear system to solve for an approximation to $\eta(x)$. Show that it yields the same algorithm as in (a). Does this tell you anything about the intermediate Newton iterates from (a)?