Math 563, Spring 2020
Last updated: 1/25/20

HOMEWORK 3 (DUE WED. FEB. 5)

Reading: Some further reading is noted below (starred is highly recommended).

- Suggested (for integrals): 9.2.3, 9.3 (omitting the proof)
- Suggested: 9.6 (Richardson extrapolation, in a bit more generality).

Code to turn in:

- The code for Q4 (trapezoidal rule; error estimate); you don't need to include the code for all the examples, just the general method. Also include the code for Q6 if completing that problem.

1. Problems (some with, some without code)

Q1 (estimating the solution). When Richardson extrapolation is not available, we can still estimate the exact solution and the error using **Aitken extrapolation** (or 'Aitken's delta-squared process').

Suppose $A(h) \approx L + O(h^p)$.

a) Assume that $O(h^p)$ is actually Ch^p for a constant C and consider

$$A(h), \quad A(h/2), \quad A(h/4).$$

Solve for L using these three evaluations of A. It is useful to first compute the quantities

$$(A(h) - A(h/2))^2, \qquad A(h) - 2A(h/2) + A(h/4).$$

b) Use the same quantities to get an estimate for p.

Q2 (rounding error). Suppose f is computed to an error of u ($\tilde{f} = f + \delta,\ |\delta| < u$) and consider the integration formula

$$\int_a^b f(x)\,dx \approx I(f) := \sum_{i=0}^{n} c_i f(x_i)$$

and the actual computed value $\tilde{I}(f)$. Assume I has a degree of accuracy at least 1.

a) Show that the 'rounding error' in $\tilde{I}(f)$ can be bounded by $(b-a)$ times something **independent** of the number of points n if the coefficients c_i are positive.

b) What is the bound when the c_i's are not all one sign?

Remark: Newton-Cotes formulas for high degree have c_i's that are not all one sign, which is one reason that they do not work well.

Q3 (Miscellaneous from class). Let $\omega_n(x) = \prod(x - x_j)$ where $x_0, \cdots, x_n$ are equally spaced in $[a, b]$ (for the closed Newton-Cotes formula).

a) (AC) Show that $\int_a^b \omega_n(x)\, dx = 0$ for even n. *Hint: don't compute this!*

b) Consider Simpson's rule

$$\int_0^1 f(s)\, ds = c_0 f(s_0) + c_1 f(s_1) + c_2 f(s_2) + \int_0^1 \cdots\, ds$$

with $s_0 = 0, s_1 = 1/2$ and $s_2 = 1$. Derive the rule using an interpolant (you don't have to simplify the error). Show that the degree of accuracy is ≥ 3.

c) Often, given points $a = x_0 < x_1 < \cdots < x_n = b$ (equally spaced), we have data $f_{i+1/2} = f(x_{i+1/2})$ at the **midpoints** $x_{i+1/2} = x_i + h/2$. Derive an appropriate composite rule and a bound for the error.

Q4 (actual behavior of the trapezoidal rule error). Use the (composite) trapezoidal rule with equally spaced points to estimate the following integrals (pick at least four).

Provide an error plot in each case using $h = (b - a)2^{-n}$ with n in an appropriate range or, if needed, some other set of better h-values. Describe the rate of convergence. Briefly discuss any interesting features and (AC) try to explain why they make sense if possible.

Suggestion: Rather than fitting, you can plot a 'reference line' of the appropriate slope to show your data has the right slope, or use the calculation in Q1. This is fine as long as the linear trend is clear!

i) $\int_0^1 \cos(1 + x^3)\, dx$ (a generic example)

ii) $\int_0^1 (x^2 - x)^4 e^x\, dx$

iii) $\int_0^{2\pi} \sin^4 x\, dx$ (exact value: $3\pi/4$)

iv) $\int_0^1 \sin(x^{1/2})\, dx$

v) $\int_0^{2\pi} \frac{1}{1+16\sin^2 x}\, dx$ (exact value: $2\pi/\sqrt{17}$)

vi) $\int_0^1 \sin(\frac{1}{(x+1/30)^2})\, dx$

Note: Do **not** compute the exact value of the integrals if they are not given!

Q5 (Richardson extrapolation without a known error series). Consider the integral

$$I = \int_0^1 \frac{e^x}{x^{1/3}}\,dx$$

Assume that its exact value cannot be found.

a) Use an appropriate rule from earlier in the homework with spacing 2^{-n} to estimate the integral. Make an error plog (log-log) and determine the rate of convergence.

b) Let $A_0(h)$ denote the approximation from (a) with spacing h. Derive a formula for $A_1(h)$ that uses A_0 at two values of h with better accuracy (one step of Richardson extrapolation). *Hint: you need the information estimated from (a) to proceed here.*

c) (AC) What is the rate of convergence for $A_1(h)$?

Q6 (Romberg integration: implementation). (AC) Let $T_0(h)$ denote the (composite) trapezoidal rule with spacing h in $[a, b]$. Recall that the trapezoidal rule has the asymptotic error series

$$T_0(h) \sim \int_a^b f(x)\,dx + \sum_{i=1}^{\infty} c_{2i} h^{2i}.$$

a) Use Richardson extrapolation and the values $h, h/2, h/4, \cdots$, to derive a set of approximations $T_1(h), T_2(h), \cdots$ of increasing order of accuracy.

b) Let $h_0 = b - a$ and $R_{i,j} = T_i(h_0 2^{-j})$ (so the coarsest approximation $T_0(h)$ is just the trapezoidal rule with one sub-interval). Write a routine `romberg(f,a,b,tol)` that generates these values as needed to obtain an approximation to the integral that has an error (estimated to be) less than `tol`.

Your code should stop once the tolerance is reached, and avoid unneeded steps (but don't spend too much time optimizing this).

Test your code on the following problems, using a tolerance of 10^{-10}. Discuss the results.

i) $\int_0^1 x e^{3\cos(x)}\,dx$

ii) $\int_0^1 \sin(x^{1/2})\,dx$