MATH 551 LECTURE NOTES
GREEN'S FUNCTIONS FOR BVPS

Topics covered

- Distributions (briefly)
  - Heaviside function
  - Dirac delta
- Green's functions (for BVPs in 1d)
  - Existence via eigenfunctions
  - Connection to Fredholm integral equations
  - Piecewise construction
- Inhomogeneous problems (with Green's functions)
  - Reciprocity (and the adjoint problem)
  - Problems with inhomogeneous BCs

1. Green's Functions (introduction)

We return to solving boundary value problems (BVPs), introducing an approach that uses **integral equations** of a sort rather than eigenfunctions. It is one of the main techniques for solving BVPs and PDEs, and plays an important role in physical problems where the associated 'kernels' are significant. The goal is to solve

$$\begin{aligned} Lu &= f(x), \quad x \in [a,b] \\ Bu &= 0 \end{aligned} \tag{H}$$

where $f(x)$ is the inhomogeneous term (here, often referred to as the **forcing**), $Bu = 0$ are typical homogeneous BCs and L is a linear differential operator

$$Lu = a_n(x)u^{(n)} + \cdots + a_1 u' + a_0 u.$$

Here, view f as the input to (H) (e.g. a pendulum being pushed) and u as the output.

Assuming a unique solution (case (A) of the FAT), we can define a 'solution operator'

$$L^{-1}f := \text{ the solution to (H) with forcing } f.$$

The superposition principle says that

$$L^{-1}f \text{ is linear (in } f).$$

This suggests that we may be able to write L^{-1} with an inner product, which turns out to be the case.[1] This gives an integral form of the solution that directly relates the input (f) to the output (u).

[1] You may recall from linear algebra that in $\mathbb{R}^n$, a linear function $f(\mathbf{x})$ from vectors to scalars can always be written as a dot product, $f(\mathbf{x}) = \mathbf{v}\cdot\mathbf{x}$ for some vector $\mathbf{v}$. This generalizes to inner products via the **Riesz representation theorem**.

Definition: A **Green's function** for the **homogeneous** boundary value problem (H) is a kernel $g(x,s)$ such that

$$u(x) = \langle G(x,s), f(s)\rangle = \int_a^b G(x,s)f(s)\,ds,$$

valid for any $f(x)$. This is an **integral representation** of the solution.
Note that $L^{-1}f := \int_a^b G(x,s)f(s)\,ds$ is a Fredholm integral operator.

1.1. **Green's function (via eigenfunctions):** The BVP can be solved with eigenfunctions, which gives the Green's function by brute force (we'll see another approach shortly). As per usual, let L and L^* have eigenfunctions ϕ_j and ψ_j and let λ_j be the adjoint eigenvalues.

Assume $\lambda \neq 0$ and set $k_j = \langle \phi_j, \psi_j\rangle$); the solution has the form

$$u = \sum_j c_j \phi_j(x).$$

Then, after manipulating $\langle Lu, \psi_j\rangle = \langle f, \psi_j\rangle$ in the standard way (see Week 2), we obtain

$$\begin{aligned} u(x) &= \textstyle\sum_j c_j\phi_j(x) \\ &= -\sum_j \frac{\langle f,\psi_j\rangle}{\lambda_j k_j}\phi_j(x) \\ &= -\sum_j \frac{1}{\lambda_j k_j}\left(\int_a^b f(s)\psi_j(s)\,dt\right)\phi_j(x) \\ &= \int_a^b \left(-\sum_j \frac{1}{\lambda_j k_j}\psi_j(s)\phi_j(x)\right) f(s)\,ds \\ &= \int_a^b g(x,s)f(s)\,ds \end{aligned}$$

where the Green's function $g(x,s)$ is given by the infinite sum

$$g(x,s) = -\sum_j \frac{\psi_j(s)\phi_j(x)}{\lambda_j k_j}. \tag{1.1}$$

Note that in the self adjoint case,

$$L^* = L \implies \psi_j = \phi_j \implies g(x,s) = g(s,x) \text{ (symmetric kernel)}.$$

Remark (connection to FIEs): The '**inverse problem**' to solve for f given u,

$$u = \int_a^b G(x,s)f(s)\,ds$$

is a Fredholm integral equation for f, with the operator self adjoint if and only if L is self-adjoint. This gives an equivalence of sorts between BVPs like (H) and FIEs.

2. Introduction (distributions)

First, some informal motivation. The Green's function $g(x,s)$, for each s, is a solution to the BVP (H) for some forcing $h(x,s)$ (to be found):

$$L_x[g(x,s)] = h(x,s).$$

Here we need notation to make clear which variable is the 'differentiated' one:

$$L_x \text{ denotes '}L \text{ acting on the } x \text{ variable'}. \tag{2.1}$$

What is h? This turns out to be subtle. We have that

$$Lu = f \implies u = \int_a^b g(x,s)f(s)\,ds.$$

Take L of both sides of the right expression to get

$$f(x) = \int_a^b L_x[g(x,s)]f(s)\,ds = \int_a^b h(x,s)f(s)\,ds.$$

It follows that h is defined by the integral equation (with an unknown kernel)

$$f(x) = \int_a^b h(x,s)f(s)\,ds \text{ for all functions } f \text{ and } x \in (a,b).$$

Now, the problem: no such function exists. The 'function' $h(x,s)$ must be zero for $x \neq s$, since it must ignore all information about f except $f(x)$. To make sense of it, we must take a detour to generalize the notion of functions and define **distributions**.

2.1. Step functions.

Define the Heaviside step function:[2]

$$H(x) = \begin{cases} 1 & x > 0 \\ 0 & x < 0 \end{cases}. \tag{2.2}$$

This simple function is quite useful in practice (and analysis). **Some notable properties:**

- The step function $H(x-s)$ is 'off' for $x < s$ then 'switches on' at $x = s$ (literally true if this is, for example, the input $H(t)$ to a circuit that is turned on at $t = 0$.)
- The function $H(x-s)$ is a 'cut-off': it sets to zero all values of the function for $x < s$:

$$H(x-s)f(x) = \begin{cases} f(x) & x > s \\ 0 & x < s \end{cases}.$$

- In an integral, $H(x-s)$ cuts off part of the integration interval:

$$\int_a^b H(x-s)f(x)\,dx = \int_s^b f(x)\,dx \text{ if } s \in (a,b)\ .$$

- The Heaviside function can be used to make a box that is 1 in an interval $[a,b]$:

$$H(x-a) - H(x-b) = \begin{cases} 1 & a < x < b \\ 0 & x < a \text{ or } x > b \end{cases}.$$

[2] The value at 0 is not relevant here; it is typically taken to be $H(0) = 1/2$ e.g. Matlab's `heaviside(x)`.

Practical note (smoothing H): Sometimes, the jump in $H(x)$ causes trouble (e.g. for numerical methods or modeling) and we need a smooth approximation. This is most easily achieved by taking

$$H_\epsilon(x) := \frac{1}{2}(1 + \tanh(x/\epsilon)).$$

This is a Heaviside with the jump smoothed to width $O(\epsilon)$ (note $H_\epsilon(x) \to x$ as $\epsilon \to 0$).

Derivative: We would also like to consider the 'derivative' of $H'(x)$ (the **Dirac delta**):

$$\delta(x) = H'(x).$$

But $H(x)$ has a jump at $x = 0$ so this is not a function, as it would have to be

$$\delta(x) = \begin{cases} 0 & x \neq 0 \\ \infty? & x = 0 \end{cases}. \quad \text{(not meaningful!)}$$

The correct way around this is to consider δ as a more general object (a **distribution**).

3. The Dirac delta

There are a few ways to get to a well-defined version of this object, which all end up in the same place. Note: the treatment here is 'practical', utilizing some informal definitions.[3]

The first approach uses integration by parts to move the derivative off of H, creating an 'equivalent' expression that does not have the problematic derivative:

$$\int \underbrace{\frac{d}{dx}H}_{\text{not defined!}} \cdot \text{smooth}\, dx = \cdots - \int \underbrace{H \cdot \frac{d}{dx}(\text{smooth})}_{\text{defined!}}\, dx$$

Let $s \in (a, b)$ and suppose f is differentiable. Then we 'compute' (really 'define')

$$\begin{aligned} \int_a^b f(x)\delta(x-s)\, dx &= \int_a^b f(x)H'(x-s)\, dx \\ &= f(x)H(x-s)\Big|_a^b - \int_a^b f'(x)H(x-s)\, dx \qquad (3.1) \\ &= f(b)H(b-s) - f(a)H(a-s) - \int_s^b f'(x)\, dx \\ &= f(b) - (f(b) - f(s)) \\ &= f(s) \end{aligned}$$

since $H(a-s) = 0$ and $H(b-s) = 1$. That is, δ can be defined as 'the derivative of' $H'(x)$ in the sense that the integration by parts (3.1) works. [4] Moreover, we have a key property:

[3] The rigorous approach is called **distribution theory**, developed first by Dirac in the 1920s and developed in full in its modern form in the 1940s by Schwartz and others.

[4] More generally, this idea can be used to define a **weak derivative**, which is a function that acts like a derivative for integration by parts. This notion is a foundational part of PDE analysis and in numerical methods (e.g. the finite element method).

Sifting property: The Dirac delta has the property that[5]

$$\int_a^b f(x)\delta(x-s)\,dx = \begin{cases} f(s) & \text{for all } s \in (a,b) \\ 0 & s < a \text{ or } s > b \end{cases}.$$

Informally: Integrating a function with $\delta(x-s)$ evaluates that function at $x = s$.

In terms of inner products: If $s \in (a,b)$ then

$$\langle f(s), \delta(x-s)\rangle = f(x) \text{ for all smooth functions } f\ .$$

Over all of $\mathbb{R}$*:* Much simpler, since there is no case work:

$$\int_{-\infty}^{\infty} f(s)\delta(x-s)\,ds = f(x) \text{ for all } x.$$

The sifting property implies a set of rules for δ that define how it behaves as an almost-function. Mostly, δ can be treated as if it were a function with the following properties:

i) **unit mass:** $\int_{-\infty}^{\infty} \delta(x)\,dx = 1$

ii) **sifting:** $\int_a^b f(x)\delta(x-s)\,dx = f(s)$ if $s \in (a,b)$

iii) **scaling (1d):** $\delta(ax) = \frac{1}{a}\delta(x)$ for $a > 0$

iv) **'derivative:'** $\delta = H'(x)$

A few further remarks are useful:

'Formal' sifting: The sifting property can also be stated in equivalent (shorthand) form as

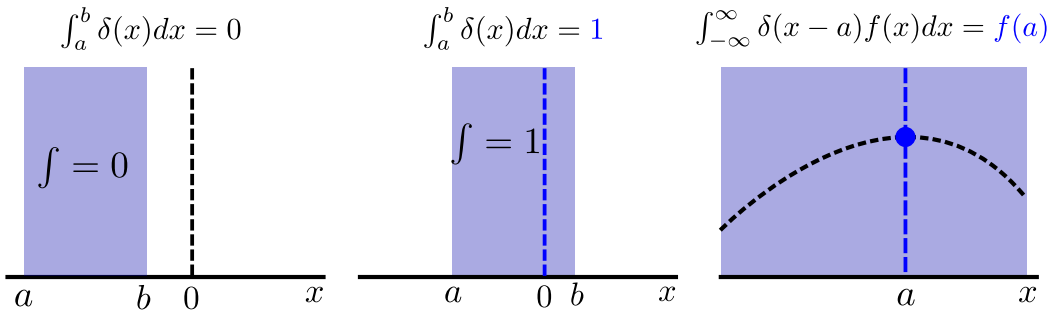
$$f(x)\delta(x-s) = f(s)\delta(x-s).$$

Addition: Deltas can also be added as if they were functions, e.g.

$$G = x + \delta(x-1) + \delta(x-2) \implies \int_0^3 f(x)G(x)\,dx = \int_0^3 x\,dx + f(1) + f(2).$$

Concentration: The first and second properties says that the entire non-zero part of the integral of $\delta(x)$ is concentrated at $x = 0$:

$$\int_a^b \delta(x)\,dx = \begin{cases} 1 & \text{if } 0 \in (a,b) \\ 0 & \text{if } 0 \notin [a,b] \end{cases}.$$



[5] If $s = a$ or $s = b$ the issue is more subtle; omitted here. It is mostly a matter of definition.

3.1. **Delta in higher dimensions:** In $\mathbb{R}^n$, $\delta(\mathbf{x})$ (with $\mathbf{x} \in \mathbb{R}^n$) works the same way:

$$\textbf{unit mass:} \quad \int_{\mathbb{R}^n} \delta(\mathbf{x})\, d\mathbf{x} = 1$$

$$\textbf{sifting:} \quad \int_{\mathbb{R}^n} f(\mathbf{x})\delta(\mathbf{x}-\mathbf{y})\, d\mathbf{x} = f(\mathbf{y}) \text{ for all } \mathbf{y} \in \mathbb{R}^n$$

$$\textbf{scaling:} \quad \delta(a\mathbf{x}) = \frac{1}{a^n}\delta(\mathbf{x}) \text{ for } a > 0$$

3.2. **Delta (limit):** We may think of the Dirac delta also as a limit of ' box' functions $\eta_\epsilon(x)$ with integral 1 concentrated in a small interval around 0. One simple set are the boxes

$$\eta_\epsilon(x) = \begin{cases} 1/(2\epsilon) & |x| < \epsilon \\ 0 & |x| > \epsilon \end{cases}.$$

We would like to say that

$$\eta_\epsilon \to \delta \text{ in some sense as } \epsilon \to 0$$

but the limit does not exist pointwise, so it is not correct to define

$$\delta(x) = \lim_{\epsilon \to 0} \eta_\epsilon(x). \text{ (wrong!).}$$

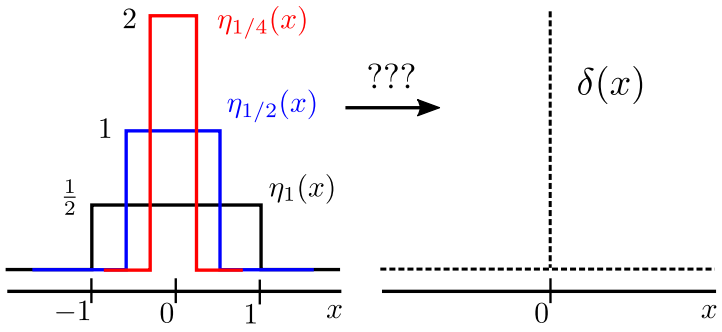
To fix this, note that the **limit works when η_ϵ is inside an integral**. It is not hard to show that if $f(x)$ is continuous then

$$\lim_{\epsilon \to 0} \left(\int_{-\infty}^{\infty} f(x)\eta_\epsilon(x-s)\, dx \right) = f(x-s)$$

which is the sifting property. Thus, this 'limit' agrees with the integration-by-parts definition. We then have a way to make the limit well-defined, by only requiring it to work in an integral:

$$\eta_e \to \delta \text{ in the sense that } \langle \eta_\epsilon, f \rangle \to \langle \delta, f \rangle \text{ for all (reasonable) } f.$$

(Technically, the formula here is 'weak convergence' of η_ϵ to δ, written $\eta_\epsilon \rightharpoonup \delta$.)



Physical view: The function $\eta_\epsilon(t)$ can represent a force applied over a short interval of time, with total momentum 1. Then $\delta(t)$ is a force with momentum 1 applied **instantly**. Even if the force is really applied over a small time, $\delta(t)$ is often a more convenient model.

3.3. **Delta (measure):** Consider a 1d object with mass density $\rho(x)$, total mass M and velocity $v(x)$. The momentum contribution from a sub-interval $[a,b]$ is

$$p = \int_a^b v(x)\rho(x)\,dx.$$

If the object is a 'point mass' located at $x=0$ with mass M then the mass density is

$$\rho = M\delta(x).$$

We then have the usual formula for momentum of a point mass with velocity v:

$$p = M\int v(x)\delta(x)\,dx = Mv(0).$$

The object $\delta(x)\,dx$ is the **Dirac measure**, which represents the density of a point mass (it is equivalent to the previous definition of δ). In this way $\delta(x)$ can be viewed as a density function whose mass (of 1) is concentrated exactly at $x=0$.

4. Piecewise Green's function

The setup: Now return to the boundary value problem

$$Lu = f,\ x\in[a,b],\ Bu = 0 \tag{4.1}$$

where L is n-th order, i.e. is has the form

$$Lu = p(x)u^{(n)} + (\text{lower order terms}). \tag{4.2}$$

and B represents n separated boundary conditions at a and b.[6]

We seek a Green's function $g(x,s)$ satisfying

$$u = \int_a^b f(s)g(x,s)\,ds \text{ (for any } f). \tag{4.3}$$

As before, let L_x denote the operator acting on the x-variable. Now equipped with $\delta(x)$, we can construct this Green's function. First, plug (4.3) into the ODE to get

$$f = Lu = \int_a^b f(s)L_x[g(x,s)]\,ds \text{ for all } f.$$

But this is exactly the sifting property, which implies that

$$L_x[g(x,s)] = \delta(x-s).$$

Similarly, plug into the BCs to get

$$0 = Bu = \int_a^b f(s)B_x[g(x,s)]\,ds \text{ for all } f \implies B_x[g(x,s)] = 0.$$

It follows that, for each $s\in(a,b)$, we seek a solution g to the boundary value problem

$$Lg = \delta(x-s),\ x\in[a,b],\ Bg = 0. \tag{4.4}$$

The idea here is that there are two pieces for $x<s$ and $x>s$ that solve the homogeneous problem (since $\delta(x-s)=0$ if $x\neq s$). Then, these two pieces are glued together in the right way so that plugging g into Lg leaves a δ from differentiating the jump at $x=s$.

[6]Note that there is no minus sign on the $p(x)$ term; we don't need the convention here because no eigenvalues are involved here, so it is easier to omit the extra minus sign.

4.1. **Derivation: construction.** Propose a Green's function in the piecewise form

$$g(x,s) = \begin{cases} g_-(x,s) & x < s \\ g_+(x,s) & x > s \end{cases}.$$

Away from the jump: Since $\delta = 0$ for $x \neq s$, both g_- and g_+ must be homogeneous solutions ($f = 0$):

$$\begin{aligned} Lg_- &= 0, \quad B_a g_- = 0, \quad x < s \\ Lg_+ &= 0, \quad B_b g_+ = 0, \quad x > s \end{aligned} \tag{4.5}$$

Let $\{w_1(x), \cdots, w_n(x)\}$ be a basis for solutions to the ODE $Lu = 0$. Then

$$g_\pm = \sum_{j=1}^{n} c_j^\pm w_j(x) \quad \text{for constants } c_j^- \text{ and } c_j^+,\ j = 1, \cdots n.$$

Thus it remains to find $2n$ equations to solve for the $2n$ unknowns. The BCs (assuming they are standard) give half of these equations:

$$Bu = 0 \implies n \text{ constraints on } g_-(a,s) \text{ and } g_+(b,s). \tag{4.6}$$

The other half will come from getting δ to appear.

At the jump (continuity): Observe that if $a(x)$ is some function then

$$\begin{aligned} a(x) \text{ has a jump at } x = s &\implies a'(x) = \text{has a } \delta(x-s) \text{ term}, \\ &\implies a''(x) \text{ has a } \delta'(x-s) \text{ term (not good!)}. \end{aligned}$$

We want only a δ to appear in Lg, and no further derivatives. It follows that

$$g \text{ must have } n-2 \text{ continuous derivatives} \tag{4.7}$$

since only $g^{(n-1)}$ and $g^{(n)}$ can have a jump and only $g^{(n)}$ must have a δ.

Jump condition: Now consider an integral of the ODE in a small interval around s:

$$\int_{s-\epsilon}^{s+\epsilon} L_x[g(x,s)]\,dx = \int_{s-\epsilon}^{s+\epsilon} \delta(x-s)\,dx = 1. \tag{4.8}$$

A jump must appear in Lg at $x = s$. The operator (4.2) can be written as

$$Lu = (p(x)u^{(n-1)})' + \text{ (lower order terms) }.$$

Plug this into the integrated equation (4.8) to get

$$\int_{s-\epsilon}^{s+\epsilon} \frac{d}{dx}\left(p(x)\frac{d^{n-1}g}{dx^{n-1}}\right) + (\text{bounded terms})\,dx = 1$$

since derivatives up to order $n-1$ can have, at worst, a jump (but never a δ). Taking $\epsilon \to 0$ shrinks the integration interval to zero, so the 'bounded' contributions vanish, leaving

$$p(x)\left(\frac{d^{n-1}g_+}{dx^{n-1}} - \frac{d^{n-1}g_-}{dx^{n-1}}\right) = 1 \text{ at } x = s. \tag{4.9}$$

The **jump condition** (4.9), continuity condition (4.7) and the boundary conditions (4.6) give the $2n$ equations needed to find the coefficients for g_- and g_+, completing the construction.

4.2. **Summary: construction.** This time, just the result. Consider the BVP

$$Lu = f,\ x \in [a,b],\ Bu = 0 \tag{4.10}$$

where $L = p(x)u^{(n)} + \cdots$ is n-th order (and the BCs are homogeneous). The Green's function $G(x,s)$ can be written in the form

$$G(x,s) = \begin{cases} g_-(x,s) & x < s \\ g_+(x,s) & x > s \end{cases}$$

where the following hold:

i) g_- and g_+ solve the **homogeneous** ODE $Lu = 0$ for $x < s$ and $x > s$, respectively

ii) g satisfies the BCs (with $g = g_-$ at $x = a$ and $g = g_+$ and $x = b$)

iii) (continuity) g_-, g_+ and their derivatives up to order $n-2$ agree at at $x = s$:

$$\frac{d^k g_+}{dx^k} - \frac{d^k g_-}{dx^k} = 0 \text{ for } k = 0, 1, \cdots n-2.$$

iv) (jump condition) the $n-1$-st derivative satisfies

$$p(x)\left(\frac{d^{n-1} g_+}{dx^{n-1}} - \frac{d^{n-1} g_-}{dx^{n-1}}\right) = 1 \text{ at } x = s.$$

The functions g_- and g_+, by (i), each have n unknown coefficients. Equation (ii) gives n conditions and (iii), (iv) provide the other n.

Procedure (Green's functions, piecewise): To construct the piecewise Green's function:

- If possible, compute a basis $w_1(x), \cdots, w_n(x)$ for solutions to the homogeneous ODE $Lu = 0$. Propose a Green's function in the form

$$g(x,s) = \begin{cases} g_-(x,s) & a \le x < s \\ g_+(x,s) & s < x \le b \end{cases}$$

  g_- and g_+ are homogeneous solutions to the ODE in $x < s$ and $x > s$:

$$g_\pm(x) = \sum_{j=1}^{n} c_j^\pm w_j(x).$$

- Apply any 'easy' BCs for g_- (at $x = a$) and g_+ (at $x = b$) to simplify (e.g. $g_-(0) = 0$).
- Use continuity/jump conditions at $x = s$ to get equations relating the c_-'s and c_+'s .
- Use any remaining BCs to get any more relations needed for c_- and c_+.
- Solve the linear system from (2)-(3) for the c's.

This procedure also solves the BVP with **homogeneous BCs**: $u = \int_a^b g(x,s)f(s)\,ds$.

Practical note: For more complicated ODEs, step (0) may not be available, in which case one can at leasts appeal to theory to guarantee they exist. Then it is a matter of getting enough information about the basis to proceed.

4.3. **Typical example (homogeneous).** A Green's function is used to solve

$$\begin{aligned} u'' + u &= f(x),\ x \in [0,1], \\ u(0) = 0,\ u'(1) &= 0 \end{aligned} \tag{4.11}$$

Let $s \in (0,1)$. Define (dropping arguments for brevity, e.g. $g = g(x,s)$)

$$g = \begin{cases} g_- & 0 \le x < s \\ g_+ & s < x \le 1 \end{cases}$$

Hom. ODE: Each piece $g_\pm$ solves $u'' + u = 0$, so we can write (see box below)

$$\begin{aligned} g_- &= a_1 \sin x + a_2 \cos x \\ g_+ &= b_1 \sin(x-1) + b_2 \cos(x-1). \end{aligned} \tag{4.12}$$

There are four unknowns to find (which will be functions of s).

Practical note: if L is linear constant coefficient, then $Lu = 0$ is **autonomous**, so if $u(x)$ is a solution then $u(x-c)$ is also a solution for any c.
When solving an ODE with a BC at $x = a$, it is useful to center the basis at $x = a$, e.g. $\sin(\pi(x-a))$ or $e^{-2(x-a)}$. The trick is limited, as the ODE must be autonomous (and linear).

BCs: Applying the homogeneous BCs to g, we get

$$\begin{aligned} 0 = g(0,s) = g_-(0,s) = a_2(s) &\implies a_2 = 0, \\ 0 = \frac{dg}{dx}(1,s) = \frac{dg_+}{dx}(0,s) = b_1(s) &\implies b_1 = 0 \end{aligned}$$

which leaves one term in each function:

$$g_-(x,s) = a(s)\sin x, \quad g_+(x,s) = b(s)\cos(x-1).$$

Jump conditions: The continuity and jump conditions determine $a(s)$ and $b(s)$. We requires g to be continuous and g' to have a jump at $x = s$:

$$\begin{aligned} g_+ = g_- \text{ at } x = s &\implies b\cos(s-1) = a\sin(s), \\ g_+' - g_-' = 1 \text{ at } x = s &\implies -b\sin(s-1) - a\cos(s) = 1. \end{aligned}$$

Solve the system: Solve the equations from the jump conditions (no more BCs to apply):

$$\begin{bmatrix} -\sin(s) & \cos(s-1) \\ -\cos(s) & -\sin(s-1) \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \implies \begin{bmatrix} a \\ b \end{bmatrix} = \frac{1}{D} \begin{bmatrix} -\sin(s-1) & -\cos(s-1) \\ \cos(s) & -\sin(s) \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

where $D = \sin(s)\sin(s-1) + \cos(s)\cos(s-1) = \cos(s-(s-1)) = \cos(1)$, so

$$a = -\cos(s-1)/\cos(1), \quad b = -\sin(s)/\cos(1).$$

The Green's function, in full, is then

$$g(x,s) = \begin{cases} -\cos(s-1)\sin x/\cos(1) & x < s \\ -\sin(s)\cos(x-1)/\cos(1) & x > s \end{cases}. \tag{4.13}$$

The solution to the BVP (4.11) is $u = \displaystyle\int_a^b g(x,s) f(s)\, ds$.

5. More on Green's functions

5.1. **Reciprocity and the adjoint:** Again, consider the homogeneous BVP

$$\begin{aligned} Lu &= f(x), \quad x \in [a,b] \\ Bu &= 0 \end{aligned} \tag{H}$$

with Green's function $g(x,s)$ and the adjoint problem

$$\begin{aligned} L^*u &= f(x), \quad x \in [a,b] \\ B^*u &= 0 \end{aligned} \tag{A}$$

with Green's function $h(x,s)$. Assume that (H) has a unique solution.

Key question: How are the Green's function's for (H) and (A) related?
Answer: As with Fredholm integral operators, we claim that

$$h(x,s) = g(s,x).$$

In particular, if L is self-adjoint then the Green's function is symmetric.

Proof: The Green's functions $g(x,s)$ and $h(x,s)$ for (H) and (A) satisfy

$$L_x[g(x,s)] = \delta(x-s), \quad B_x g = 0, \tag{5.1}$$

$$L_x^*[h(x,s)] = \delta(x-s), \quad B_x^* h = 0. \tag{5.2}$$

Here, we use an integral instead of $\langle \cdot, \cdot \rangle$ for clarity (pay attention to the integration variable!). Let v solve the adjoint problem (A); then by definition of the Green's function,

$$v(x) = \int_a^b h(x,s) f(s)\, ds. \tag{5.3}$$

Now take the inner product of (5.1) with v (in x):

$$\begin{aligned} \delta(x-s) &= L_x[g(x,s)] \\ &\downarrow \\ \int_a^b \delta(x-s) v(x)\, dx &= \int_a^b L_x[g(x,s)] v(x)\, dx && \text{(by def'n of } g \text{ (5.1))} \\ &= \int_a^b g(x,s) L_x^* v\, dx && \text{(adjoint property)} \\ &= \int_a^b g(x,s) f(s)\, ds && (v \text{ solves adj. problem}) \end{aligned}$$

There are no boundary terms since the BCs are homogeneous. By the sifting property,

$$v(s) = \int_a^b g(x,s) f(x)\, dx \implies v(x) = \int_a^b g(s,x) f(s)\, ds$$

after swapping s and x for notation. But this is just (5.3) (the definition of the Green's function for the adjoint problem), which completes the proof.

Note: This result is a **reciprocity theorem**; see the discussion in the textbook on Maxwell's reciprocity theorem (Chapter 9) for details.

Summary (reciprocity/adjoint): Let $g(x,s)$ be the Green's function for the problem (H) (with homogeneous BCs). Then:

- The Green's function for the adjoint problem (A) is $g(s,x)$.
- If L is self-adjoint then the Green's function is symmetric ($g(x,s) = g(s,x)$).
- L acts on the first argument of g and L^* acts on the **second** argument of g as follows:

$$\begin{aligned} L_x[g(x,s)] &= \delta(x-s) \\ L_s^*[g(x,s)] &= \delta(s-x) \end{aligned} \tag{5.4}$$

  This comes from the result $L_x^* g(s,x) = \delta(x-s)$ with variables swapped.

5.2. **Revisiting the example.** Now consider the example with **inhomogeneous BCs**,

$$\begin{aligned} u'' + u &= f(x),\ x \in [0,1], \\ u(0) = p,\ u'(1) &= q \end{aligned} \tag{5.5}$$

We first consider **homogeneous** BCs to find the Green's function, then use the adjoint property and integration by parts (with boundary terms!) to deal with the **inhomogeneous** BCs.

Part I (Green's function). This was done in subsection 4.3. The result:

$$g(x,s) = \begin{cases} g_- & 0 \le x < s \\ g_+ & s < x \le 1 \end{cases} \implies g(x,s) = \begin{cases} -\cos(s-1)\sin x/\cos(1) & x < s \\ -\sin(s)\cos(x-1)/\cos(1) & x > s \end{cases}.$$

The solution to the problem with homogeneous BCs ($p = q = 0$) is

$$u(x) = \int_a^b g(x,s) f(s)\, ds.$$

Part II (Solve the BVP). Now return to the full problem (5.5). Let $Lu = u'' + u$.

Following the work in subsection 5.1, Take the inner product with $g(x,s)$ (carefully):

$$\begin{aligned} \int_0^1 f(s) g(x,s)\, ds &= \int_0^1 L[u(s)] g(x,s)\, ds \\ &= \text{bdry terms} + \int_0^1 u(s) L_s[g(x,s)]\, ds \\ &= \text{bdry terms} + \int_0^1 u(s)\delta(s-x)\, ds. \end{aligned}$$

The boundary terms are obtained from IBP twice:

$$\begin{aligned} \int_0^1 (u''(s) + u(s)) g(x,s)\, ds &= (u'g - ug_s)\Big|_{s=0}^{s=1} + \int_0^1 u(g_{ss} + g)\, ds \\ &= \text{bdry terms} + \int_0^1 u L_s[g(x,s)]\, ds. \end{aligned}$$

Finally, simplify via the sifting property to get

$$u(x) = \int_0^1 g(x,s)f(s)\,ds - (u'g - ug_s)\Big|_{s=0}^{s=1}.$$

Evaluating the BC term for (5.5),

$$\begin{aligned} u(x) &= \int_0^1 g(x,s)f(s)\,ds - u'(1)g_-(x,1) + u(0)(g_+)_s(x,0) \\ &= \int_0^1 f(s)g(x,s)\,ds + q\frac{\sin x}{\cos 1} - p\frac{\cos(x-1)}{\cos 1}. \end{aligned}$$

This has the form

$$\begin{aligned} u(x) &= \int_0^1 g(x,s)f(s)\,ds + \left(u\frac{\partial g}{\partial s}(x,s) - \frac{du}{dx}g(x,s)\right)\Big|_0^1 \\ &= \quad \text{response to } f \quad + \qquad \text{response to BCs.} \end{aligned}$$

The second term is a 'boundary' Green's function analogous to the first term, but over the boundary of the domain (here, it is not so easy to see because the boundary is zero-dimensional). The same pattern will be found in higher dimensions.

Summary (Green's function with inhomogeneous BCs): Consider

$$Lu = f, \quad Bu = \mathbf{c}.$$

As with eigenfunctions, there **two parts** to the solution procedure.

Part I: Obtain the Green's function for the problem with **homogeneous BCs**:

$$Lg = \delta(x-s), \qquad Bg = 0.$$

If the problem has homogeneous BCs, we are done ($u(x) = \int_a^b g(x,s)f(s)\,ds$).

Part II (full): Take the inner product with g (here s is the integration variable):

$$\int_a^b L[u(s)]g(x,s)\,ds = \int_a^b f(s)g(x,s)\,ds.$$

Now integrate by parts on the left term to get

$$\text{bdry terms} + \langle u(s), L_s^*[g(x,s)]\rangle = \langle f(s), g(x,s)\rangle$$

and then the rule (5.4) to get

$$\begin{aligned} \text{bdry terms} + u(x) &= \langle f(s), g(x,s)\rangle \\ \implies u(x) &= \int_a^b f(x)g(x,s)\,ds - \text{bdry terms}. \end{aligned} \tag{5.6}$$

Part II (shortcut): Simply compute the 'boundary terms' that arise from

$$\int_a^b L[u(s)]g(x,s)\,ds = (\text{bdry terms}) + \int_a^b u(s)L_s^*[g(x,s)]\,ds$$

and plug them directly into (5.6) (skipping the 'derivation' of Part II).

5.3. **Example (zero eigenvalue).** When there is a zero eigenvalue, the Green's function construction fails to work: there is no $g(x,s)$ that gives a unique solution for all f, since the solution is not unique. For example, consider

$$\begin{aligned} u'' + u &= f, \quad x \in [0,\pi] \\ u'(0) &= u'(L) = 0 \end{aligned} \tag{5.7}$$

Note that $\lambda = 0$ is an eigenvalue of $Lu = u'' + u$ with eigenfunction $\phi = \cos x$. Alternatively, $\lambda = -1$ is an eigenvalue of $Lu = u''$ (then the $+u$ shifts it to zero). By the Fredholm alternative,

$$\text{a solution exists iff } 0 = \langle f, \phi_0 \rangle = \int_0^1 f(x) \cos x \, dx. \tag{5.8}$$

When a solution exists, it is not unique; there is an arbitary multiple of ϕ_0. Clearly, we cannot represent the solution for all f as

$$u(x) = c\phi_0 + \int_0^1 g(x,s) f(s) \, ds$$

since this fails to account for the condition (5.8). For the (small) fix, see the next section.

Suppose we try to construct the Green's function directly (note: this is an excessive way to show the solution does not always exist!). We have

$$g(x,s) = \begin{cases} g_- & 0 \le x < s \\ g_+ & s < x \le 1 \end{cases}$$

where (after using the BCs)

$$g_- = a \cos x, \quad g_+ = b \cos(x - \pi).$$

Applying the continuity/jump conditions,

$$\begin{aligned} g_+ - g_- = 0 \text{ at } x = s &\implies a\cos(s) = b\cos(s-\pi) \\ \frac{dg_+}{dx} - \frac{dg_-}{dx} = 1 \text{ at } x = s &\implies -a\sin(s) - b\sin(s-\pi) = 1 \end{aligned}$$

so the linear system to solve is

$$\begin{bmatrix} \cos(s) & -\cos(s-\pi) \\ -\sin(s) & -\sin(s-\pi) \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

The determinant of the matrix is

$$D = -\cos(s)\sin(s-\pi) - \sin(s)\cos(s-\pi) = -\sin(s - (s-\pi)) = -\sin\pi = -0$$

so there is no solution for a and b; the construction fails. A modified procedure must be used to avoid the problem (see below).

5.4. **Generalized Green's functions (zero eigenvalue case).** The presence of a zero eigenvalue does not invalidate the Green's function method - it just needs to be modified to **'remove' the zero eigenvalue part**. Suppose L is self-adjoint, say,

$$Lu = (p(x)u')' + q(x)u$$

and we have the problem (with separated BCs)

$$\begin{gathered} Lu = f(x), \quad x \in [a,b] \\ Bu = 0 \\ \lambda = 0 \text{ is an eig. of } L \text{ (with } Bu = 0) \end{gathered} \tag{H_0}$$

Let ϕ_0 be the eigenfunction for $\lambda = 0$. Recall that by the Fredholm alternative,

$$u(x) = \begin{cases} \text{no soln.} & \text{if } \langle f, \phi_0 \rangle \neq 0 \\ u_p(x) + c\phi_0(x) & \text{if } \langle f, \phi_0 \rangle = 0 \end{cases}$$

where u_p is a 'particular' solution (the part orthogonal to ϕ_0) and c is arbitrary.

We may modify the problem by removing the bad ϕ_0 component from f to get

$$Lu = f - \frac{\langle f, \phi_0 \rangle}{\langle \phi_0, \phi_0 \rangle}\phi_0(x). \tag{H_m}$$

Now (H_m) (with the same BCs) always has a solution

$$u(x) = u_p(x) + c\phi_0(x)$$

and the particular part $u_p(x)$ is **unique** up to a multiple of ϕ_0.

This modified problem can be solved using a **generalized Green's function** g_m. Note that plugging $f = \delta(x-s)$ into the modified problem we have

$$\langle \delta(x-s), \phi_0 \rangle = \phi_0(x-s)$$

by the sifting property. Thus $g_m(x,s)$ should be found by solving

$$L_x[g_m(x,s)] = \delta(x-s) - \frac{1}{\langle \phi_0, \phi_0 \rangle}\phi_0(x-s)\phi_0(x) \tag{5.9}$$

with the given BCs. Essentially, we have removed the ϕ_0 component of $\delta(x-s)$ to allows this problem to be solved. In terms of this new Green's function, the solution to the modified problem (H_m) is (for arbitrary c)

$$u(x) = \int_a^b g_m(x,s)f(s)\,ds + c\phi_0(x).$$

It is also the Green's function for the original problem in the sense that

$$u(x) = c\phi_0 + \int_a^b g_m(x,s)f(s)\,ds \text{ solves } (H_0) \text{ if } \langle f, \phi_0 \rangle = 0.$$

The process for constructing g_m is the same as before, except there is an arbitrary constant that appears. This is not a concern, however, since the solution $u(x)$ is only unique up to an arbitrary multiple of ϕ_0. For details, see Section 9.4 of Haberman.

6. A note on the heat equation

Return to a typical IBVP for the heat equation,

$$\begin{aligned} u_t = u_{xx} + h(x,t), \quad x \in [0,\pi],\, t > 0 \\ u(0,t) = u(\pi,t) = 0 \\ u(x,0) = f(x) \end{aligned}$$

with $\phi_n = \sin(nx)$ and $\lambda_n = n^2$ and the solution is

$$u(x,t) = \sum_{n=1}^{\infty} c_n(t)\phi_n(x), \quad c_n(t) = f_n e^{-\lambda_n t} + e^{-\lambda_n t}\int_0^t e^{\lambda_n \tau} h_n(\tau)\, d\tau,$$

$$f = \sum_{n\geq 1} f_n \phi_n, \quad h(x,t) = \sum_{n\geq 1} g_n(t)\phi_n(x).$$

Also, noting that $\langle \phi_n, \phi_n \rangle = \pi/2$,

$$f_n = \frac{2}{\pi}\int_0^\pi f(x)\phi_n(x)\, dx, \quad h_n = \frac{2}{\pi}\int_0^\pi h(x,t)\phi_n(x)\, dx.$$

Plugging this in and rearranging, we get

$$\begin{aligned} u(x,t) &= \sum_{n=1}^{\infty}\left(f_n e^{-\lambda_n t} + e^{-\lambda_n t}\int_0^t e^{\lambda_n \tau} h_n(\tau)\, d\tau \right)\phi_n(x), \\ &= \frac{2}{\pi}\sum_{n=1}^{\infty} e^{-\lambda_n t}\int_0^\pi f(s)\phi_n(s)\phi_n(x)\, ds + \frac{2}{\pi}\sum_{n=1}^{\infty}\int_0^t\int_0^\pi e^{-\lambda_n(t-\tau)} h(s,\tau)\phi_n(s)\phi_n(x)\, ds\, d\tau \\ &= \int_0^\pi G(x,s,t) f(s)\, ds + \frac{2}{\pi}\int_0^t \left(\int_0^\pi G(x,s,t-\tau) h(s,\tau)\, ds \right) d\tau. \end{aligned}$$

for the 'Green's function'

$$G(x,s,t) = \frac{2}{\pi}\sum_{n=1}^{\infty} e^{-\lambda_n t}\phi_n(x)\phi_n(s)$$

This is a Green's function of sorts of the heat equation (with a t-dependence); the linear dependence of the solution on the initial condition and source is evident.

Duhamel's principle: For the right hand side, note that the effect of the source is an integral over solutions to the homogeneous problem. Precisely,

$$\text{source part } = \int_0^t \tilde{u}(x,s;\tau)\, d\tau, \quad \tilde{u}(x,s;\tau) = \int_0^\pi G(x,s,t-\tau) h(s,\tau)\, ds$$

where the $\tilde{u}$'s solve, for each τ, the IBVP with the source 'frozen' as an IC at time τ:

$$\begin{aligned} \tilde{u}_t = \tilde{u}_{xx}, \quad x \in [0,\pi],\, t > \tau \\ \tilde{u}(0,t) = \tilde{u}(\pi,t) = 0 \\ \tilde{u}(x,\tau) = h(x,\tau) \end{aligned}$$

The result here is **Duhamel's principle**: the input to the system at each time propagates forward in time and the solution is the superposition of these effects.