# MATH 551 LECTURE NOTES
# FREDHOLM INTEGRAL EQUATIONS
# (A BRIEF INTRODUCTION)

## Topics covered

- Fredholm integral operators
  - Integral equations (Volterra vs. Fredholm)
  - Eigenfunctions for separable kernels
  - Adjoint operator, symmetric kernels
- Solution procedure (separable)
  - Solution via eigenfunctions (first and second kind)
  - Shortcuts: undetermined coefficients
  - An example (separable kernel, $n = 2$)
- Non-separable kernels (briefly)
  - Hilbert-Schmidt theory

## Preface

Read the **Fredholm alternative** notes before proceeding. This is covered in the book (Section 9.4), but the material on integral equations is not. For references on integral equations (and other topics covered in the book too!), see:

- Riley and Hobson, *Mathematical methods for physics and engineering* (this is an extensive reference, also for other topics in the course)
- Guenther and Lee, *Partial differential equations of mathematical physics and integral equations* (more technical; not the best first reference)
- J.D. Logan, *Applied mathematics* (more generally about applied mathematics techniques, with a good section on integral equations)

## 1. Fredholm integral equations: introduction

Differential equations $Lu = f$ are a subset of more general equations involving linear operators $L$. Here, we give a brief treatment of a generalization to **integral equations**.

To motivate this, every ODE IVP can be written as an 'integral equation' by integrating. For instance, consider the first order IVP

$$\frac{du}{dx} = f(x, u(x)), \quad u(a) = u_0. \tag{1.1}$$

Integrate both sides from $a$ to $x$ to get the **integral equation**

$$u(x) = u_0 + \int_a^x f(s, u(s))\, ds. \tag{1.2}$$

If $u$ solves (1.2) then it also solves (1.1); they are 'equivalent' in this sense. However, it is slightly more general as $u$ does *not* need to be differentiable.

**Definition:** A **Volterra integral equation** for $u(x)$ has the form

$$u(x) = a + \int_a^x k(x,t)u(t)\,dt.$$

The function $k(x,t)$ is the **kernel**. Note the integral upper bound is $x$ (the ind. variable).

Volterra integral equations are 'equivalent' to ODE **initial value problems** on $x \geq a$ for linear ODEs. They will not be studied here.

In contrast, ODE **boundary value problems** generalize to **Fredholm** integral equations. Such an equation involves an integral over the whole domain (not up to $x$):

**Definition:** A **Fredholm integral equation** (FIE) has two forms:

- **First kind:** (FIE-1)

$$\int_a^b k(x,t)u(t)\,dt = f(x) \tag{1.3}$$

- **Second kind:** (FIE-2)

$$\gamma u(x) + \int_a^b k(x,t)u(t)\,dt = f(x) \tag{1.4}$$

The 'second kind' is the first kind plus a multiple of $u$. Note that the operator

$$Lu = \gamma u + \int_a^b k(x,t)u(t)\,dt$$

is **linear**; this is called a **Fredholm integral operator**.

The simplest kernels are **separable** ('degenerate'), which have the form

$$k(x,t) = \sum_{j=1}^{n} \alpha_j(x)\beta_j(t) = \textbf{finite} \text{ sum of 'separated' products.} \tag{1.5}$$

More complicated kernels are **non-separable**. Examples include:

- $k = 1/(x-t)$ (Hilbert transform)
- $k = e^{-ixt}$ (Fourier transform)
- $k = e^{-xt}$ (Laplace transform)

We will study the last two later. Writing a non-separable kernel in the form (1.5) would require an **infinite** series, which complicates the analysis. Instead, we will utilize an integral form and some complex analysis later to properly handle non-separable kernels.

## 2. Solving separable FIEs

Suppose $k$ is separable and $L$ is a Fredholm integral operator of the first kind:

$$Lu = \int_a^b k(x,t)u(t)\,dt, \qquad k(x,t) = \sum_{j=1}^n \alpha_j(x)\beta_j(t). \tag{2.1}$$

We wish to solve the **eigenvalue problem**

$$L\phi = \lambda\phi.$$

This can always be done for a separable kernel. The main result is the following:

**Eigenfunctions for FIE-1:** For the FIE (2.1) with separable kernel,

1) there are $n$ non-zero eigenvalues $\lambda_1, \cdots, \lambda_n$ with eigenfunctions $\phi_1, \cdots, \phi_n$. Each eigenfunction is a linear combination of the $\alpha_j$'s, i.e. $\phi_j(x) = \sum c_{jk}\alpha_k(x)$
2) $\lambda^\infty = 0$ is an eigenvalue with infinite multiplicity: an infinite set of (orthogonal) eigenfunctions $\phi_m^\infty$ $(m = 1, 2, \cdots)$ characterized by

$$0 = \langle \phi_m^\infty, \beta_j \rangle = \int_a^b \phi_m^\infty(t)\beta_j(t)\,dt \text{ for } j = 1, 2, \cdots n. \tag{2.2}$$

The eigenfunctions from (1) and (2) together are a basis for $L^2[a,b]$.

**Proof, part 1 (non-zero eigenvalues):** Consider $\lambda \neq 0$; look for an eigenfunction

$$\phi = \sum_{j=1}^n c_j\alpha_j.$$

**Derivation:** Plug in this expression into the operator $L\phi = \int_a^b k(x,t)\phi(t)\,dt$:

$$\begin{aligned} L\phi &= \int_a^b \left( \sum_{i=1}^n \alpha_i(x)\beta_i(t) \right) \sum_{j=1}^n c_j\alpha_j(t)\,dt \\ &= \sum_{1\le i,j\le n} c_j\alpha_i(x) \left( \int_a^b \beta_i(t)\alpha_j(t)dt \right) \\ &= \sum_{i,j} A_{ij}c_j\alpha_i(x) \end{aligned}$$

where $A_{ij} = \int_a^b \beta_i(t)\alpha_j(t)\,dt$. Now set this equal to

$$\lambda\phi = \sum_{i=1}^n c_i\alpha_i(x).$$

The coefficients on $\alpha_i$ must match, leaving the linear system

$$A\mathbf{c} = \lambda\mathbf{c}, \quad \mathbf{c} = (c_1, \cdots, c_n)^T, \quad A_{ij} = \int_a^b \beta_i(t)\alpha_j(t)\,dt. \tag{2.3}$$

so the eigenvalue problem for $L$ is equivalent to a **matrix eigenvalue problem** for an $n \times n$ matrix. Assuming $A$ is invertible, the result follows.

**Claim 2 (zero eigenvalue):** We show that $\lambda = 0$ is always an eigenvalue with **infinite** multiplicity. Observe that

$$L\phi = \sum_{j=1}^{n} \alpha_j(x) \left( \int_a^b \beta_j(t)\phi(t)\, dt \right) = \sum_{j=1}^{n} \langle \beta_j, \phi \rangle \alpha_j(x).$$

This must be true for all $x$, so it follows that

$$L\phi = 0 \iff \langle \beta_j, \phi \rangle = 0 \text{ for } j = 1, \cdots, n.$$

That is, the set of eigenfunctions for $\lambda = 0$ is precisely the space of functions orthogonal to the span of the $\beta_j$'s. But $L^2[a, b]$ is infinite dimensional and the span of $\{\beta_j\}$ has dimension $n$, so the orthogonal complement is infinite dimensional - this proves the claim. That is,

$$L^2[a, b] = \text{span}(\{\beta_j\}) + \{L\phi = 0\}.$$

The fact that the basis for $\{L\phi = 0\}$ is countable (a sequence indexed by $n$) also follows from the fact that $L^2[a, b]$ has this property.[1] The eigenfunctions $\phi_m^\infty$ can be constructed using Gram-Schmidt (see example in subsection 2.1).

2.1. **Example (eigenfunctions).** Define the (separable) kernel

$$\begin{gathered} k(x, t) = 4xt - 5x^2t^2, \\ (\alpha_1 = x,\ \alpha_2 = x^2,\ \beta_1 = 4t,\ \beta_2 = -5t^2) \end{gathered} \tag{2.4}$$

and consider the FIE of the first kind

$$Lu := \int_0^1 k(x, t)u(t)\, dt = f.$$

The result says $L$ has $n = 2$ non-zero eigenvalues. Look for an eigenfunction

$$\phi = c_1\alpha_1 + c_2\alpha_2 \tag{2.5}$$

then plug in and integrate to get

$$\int_0^1 (\alpha_1(x)\beta_1(t) + \alpha_2(x)\beta_2(t))\,(c_1\alpha_1(t) + c_2\alpha_2(t))\, dt = \lambda(c_1\alpha_1(x) + c_2\alpha_2(x))$$

which gives (equating coefficients of $\alpha_1(x), \alpha_2(x)$ on the LHS/RHS as in (2.3))

$$\begin{bmatrix} \int_0^1 \beta_1\alpha_1\, dt & \int_0^1 \beta_1\alpha_2\, dt \\ \int_0^1 \beta_2\alpha_1\, dt & \int_0^1 \beta_2\alpha_2\, dt \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \lambda \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}.$$

Plugging in the values from (2.4), the eigenvalue problem is

$$A\mathbf{c} = \lambda\mathbf{c}, \quad A = \begin{bmatrix} 4/3 & 1 \\ -5/4 & -1 \end{bmatrix} \tag{2.6}$$

$$\implies \lambda = 1/2, \mathbf{c} = (-6, 5)^T \quad \lambda = -1/6, \mathbf{c} = (2, -3)^T.$$

Translating back to the integral equation with $\phi$ given by (2.5), the non-zero $\lambda$'s and $\phi$'s are

$$\{\lambda_1 = 1/2,\ \phi_1 = -6x + 5x^2\}, \quad \{\lambda_2 = -1/6,\ \phi_2 = 2x - 3x^2\}$$

along with the zero eigenvalue and eigenfunctions $\{\lambda^\infty = 0,\ \phi_1^\infty, \phi_2^\infty, \cdots\}$.

[1] In functional analysis terms, it is a 'separable Hilbert space'.

**Zero eigenvalue (details):** For $\lambda = 0$, use the characterization (2.2):

$$L\phi = 0 \iff \langle \phi, \beta_1 \rangle = \langle \phi, \beta_2 \rangle = 0 \iff \int_0^1 t\phi(t)\, dt = \int_0^1 t^2 \phi(t)\, dt = 0.$$

First, we need to orthogonalize. Define $p_1 = t$ and

$$p_2 = t^2 - \frac{\langle t^2, t \rangle}{\langle t, t \rangle} t = t^2 - \frac{3}{4} t.$$

It follows that if $g(t)$ is any function then a zero eigenfunction is

$$\phi^\infty = g - \sum_{j=1}^{2} \frac{\langle g, p_j \rangle}{\langle p_j, p_j \rangle} p_j.$$

The whole set can be found using Gram-Schmidt, e.g. with $p_3 = 1, p_4 = t^3, \cdots$. For instance,

$$\phi = 1 - \sum_{j=1}^{2} \frac{\langle 1, p_j \rangle}{\langle p_j, p_j \rangle} p_j = 1 - \frac{3}{2} t - \frac{-1/24}{1/80}(t^2 - \frac{3}{4} t) = 1 - 4t + \frac{10}{3} t^2$$

is an eigenfunction for $\lambda = 0$ (orthogonal to $t$ and $t^2$).

2.2. **Adjoint operator.** The operator $L$ is not self-adjoint in general. To solve FIEs with an eigenfunction expansion, we need the adjoint $L^*$ and its eigenfunctions $\psi_n$. To obtain it, let $\langle \cdot, \cdot \rangle$ denote the $L^2$ inner product $\langle f, g \rangle = \int_a^b f(x) g(x)\, dx$.

We look for an integral operator $L^*$ such that

$$\langle Lu, v \rangle = \langle u, L^* v \rangle \text{ for all } u, v \in L^2[a, b]. \tag{2.7}$$

To find the adjoint, carefully exchange integration order (**pay attention to** which of $t$ or $x$ is the 'integration variable' here!). Compute

$$\begin{aligned} \langle Lu, v \rangle &= \int_a^b \left( \int_a^b k(x,t) u(t)\, dt \right) v(x)\, dx \\ &= \int_a^b \int_a^b k(x,t) u(t) v(x)\, dt\, dx \\ &= \int_a^b \left( \int_a^b k(x,t) v(x)\, dx \right) u(t)\, dt \\ &= \langle u, L^* v \rangle \end{aligned}$$

Note the inner integration variable above is now $x$ and not $t$. From this result, we see that

$$(L^* v)(t) = \int_a^b k(x,t) v(x)\, dx \implies (L^* v)(x) = \int_a^b k(t,x) v(t)\, dt$$

after swapping symbols for $t$ and $x$. Comparing to $Lu = \int_a^b k(x,t) u(t)\, dt$, we have the following characterization for the adjoint:

**FI operator adjoint:** The FI operator of the first kind and its adjoint are

$$Lu = \int_a^b k(x,t)u(t)\,dt, \quad L^*v = \int_a^b k(t,x)v(t)\,dt \tag{2.8}$$

i.e. the adjoint has kernel $k(t,x)$ (the kernel of $L$ with the variables swapped). Moreover,

$$L = L^* \iff k(x,t) = k(t,x).$$

That is, $L$ is self-adjoint if and only if the kernel is **symmetric** (e.g. $k(t,x) = e^{t+x}$).

## 3. Solving FIEs of the first kind (separable)

We are now equipped to solve

$$Lu = f \tag{3.1}$$

where $L$ has a separable kernel (1.5). Note that from (2.8), it follows that $L^*$ also has a separable kernel.

Thus the adjoint $L^*$ also has $n$ non-zero eigenvalues with eigenfunctions $\psi_1, \cdots, \psi_n$ (same structure). Moreover, the $\psi$'s and $\phi$'s are **bi-orthogonal**. That is, $\langle \phi_j, \psi_k \rangle = 0$ for $j \neq k$ and the same for the $\phi_m^\infty$'s and $\psi_m^\infty$'s and between the two sets (e.g. $\langle \phi_j, \psi_m^\infty \rangle = 0$ for all $j, m$).

**Reminder:** to get the coefficient of $\phi_j$, take the $L^2$ inner product with $\psi_j$, e.g.

$$f = \sum_j c_j \phi_j \implies \langle f, \psi_j \rangle = c_j \langle \phi_j, \psi_j \rangle.$$

**Solving the FIE:** Project by taking $\langle \cdot, \psi_j \rangle$ to get

$$\langle Lu, \psi_j \rangle = \langle f, \psi_j \rangle.$$

We need to write $Lu$ and $f$ in terms of the eigenfunctions.

- *For the RHS ($f$):* This is direct:

$$f = \sum_j f_j \phi_j + \sum_m f_m^\infty \phi_m^\infty. \tag{3.2}$$

Let $k_j = \langle \phi_j, \psi_j \rangle$. Take the inner product with $\psi_j$ to get

$$f_j = \langle f, \psi_j \rangle / k_j.$$

The other coefficients will only be needed later.

- *For the LHS ($Lu$):* First write

$$u = \sum_j c_j \phi_j + \sum_m c_m^\infty \phi_m^\infty. \tag{3.3}$$

The second term is sent to zero by $L$:

$$Lu = \sum_j c_j L\phi_j = \sum_j \lambda_j c_j \phi_j.$$

Taking the inner product with $\psi_j$, we get

$$\langle Lu, \psi_j \rangle = \lambda_j c_j \langle \phi_j, \psi_j \rangle.$$

• *Combining:* Equating the two parts, we find that

$$c_j = f_j = \langle f, \psi_j \rangle / k_j, \quad j = 1, \cdots, n.$$

**Solvability:** What about the $c_m^\infty$'s? The adjoint $L^*$ has a zero eigenvalue, so we are in FAT case (B). A solvability condition decides between no solution or infinitely many solutions.

To find this condition, project onto one a eigenfunction for $\lambda = 0$, i.e. take the inner product of $Lu = f$ with one of the $\psi_m^\infty$'s:

$$\langle f, \psi_m^\infty \rangle = \langle Lu, \psi_m^\infty \rangle = \langle u, L^* \psi_m^\infty \rangle = 0.$$

Thus a solution exists **only when** $f(x)$ has no $\phi_m^\infty$ component for any $m$, i.e. if and only if

$$f = \sum_{j=1}^{n} f_j \phi_j. \tag{3.4}$$

**Summary of solution (separable FIE-1):** If $f$ is of the form (3.4), the solution is

$$u = \sum_{j=1}^{n} c_j \phi_j + \sum_m c_m^\infty \phi_m^\infty, \quad c_j = \frac{\langle f, \psi_j \rangle}{\langle \phi_j, \psi_j \rangle}$$

and the $c_m^\infty$'s are arbitrary. If $f$ is not of the form (3.4) then there is no solution.

3.1. **The shortcut.** There is a way to get around the full expansion. Recall that

$$\phi_j(x) = \sum_\ell d_{\ell,j} \alpha_\ell(x)$$

i.e each eigenfunction (for non-zero $\lambda$) is a linear combination of the $\alpha$'s. Then

$$Lu = \sum_j \lambda_j c_j \phi_j = \sum_j \lambda_j c_j \left( \sum_\ell d_{\ell,j} \alpha_\ell(x) \right) = \sum_\ell C_\ell \alpha_\ell(x).$$

It follows that we could anticipate the solution and guess

$$u = \sum_j C_j \alpha_j$$

knowing that the result for $u$ will be in this form. We **lose the bi-orthogonal structure** (no eigenfunctions), but the $\phi$'s are not needed to calculate the solution.
If $n$ is small, the shortcut is much easier (the linear system is small).

3.2. **Example:** Return to the previous example (subsection 2.1),

$$Lu := \int_0^1 k(x,t) u(t)\, dt = f,$$

with separable kernel

$$k(x,t) = 4xt - 5x^2t^2, \qquad (\alpha_1 = x,\ \alpha_2 = x^2,\ \beta_1 = 4t,\ \beta_2 = -5t^2),$$

and non-zero eigenvalues/functions

$$\{\lambda_1 = 1/2,\ \phi_1 = -6x + 5x^2\}, \quad \{\lambda_2 = -1/6,\ \phi_2 = 2x - 3x^2\}.$$

along with the zero eigenvalue and eigenfunctions $\{\phi_m^\infty\}$. Suppose the problem is

$$Lu = 8x - 7x^2.$$

Note that $\lambda = 0$ is always an eigenvalue, so we are in case (B) of the FAT. Here, the RHS is in the span of $\phi_1$ and $\phi_2$ (both linear combinations of $x$ and $x^2$), so there is a solution. It is unique up to an arbitrary sum of the form $\sum_m c_m^\infty \phi_m^\infty$.

**Direct method:** Note that the kernel is symmetric here. Thus $L$ is self-adjoint and $\psi_j = \phi_j$. Write the solution as

$$u = a_1\phi_1 + a_2\phi_2 + \sum_m c_m^\infty \phi_m^\infty.$$

Take the inner product of the equation with $\phi_j$:

$$\langle f, \phi_j\rangle = \langle Lu, \phi_j\rangle = \langle u, L^*\phi_j\rangle = \lambda_j a_j \langle \phi_j, \phi_j\rangle \implies a_j = \frac{\langle f, \phi_j\rangle}{\langle \phi_j, \phi_j\rangle}.$$

The coefficients can be computed from here; then

$$u = a_1\phi_1 + a_2\phi_2 + \sum_m c_m^\infty \phi_m^\infty$$

for arbitrary $c_m^\infty$'s (details not carried out here).

**Shortcut method:** Instead look for the 'unique part' as a linear combination of $\alpha$'s:

$$u = b_1\alpha_1(x) + b_2\alpha_2(x).$$

Plug into the equation and write as a linear system:

$$\begin{bmatrix} \int_0^1 \beta_1\alpha_1\,dt & \int_0^1 \beta_1\alpha_2\,dt \\ \int_0^1 \beta_2\alpha_1\,dt & \int_0^1 \beta_2\alpha_2\,dt \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} 8 \\ -7 \end{bmatrix} \implies \begin{bmatrix} 4/3 & 1 \\ -5/4 & -1 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} 8 \\ -7 \end{bmatrix}$$

so $b_1 = 12$ and $b_2 = -8$. The solution is then (for arbitrary $c_m^\infty$'s)

$$u(x) = 12x - 8x^2 + \sum_m c_m^\infty \phi_m^\infty.$$

## 4. FIEs of the second kind (still separable)

Now consider the FI operator of the second kind,

$$Lu = L_1 u + \gamma u \tag{4.1}$$

where $L_1$ is an FIE of the first kind with a separable kernel:

$$L_1 u = \int_a^b k(x,t)u(t)\,dt, \quad k(x,t) = \sum_{j=1}^n \alpha_j(x)\beta_j(t).$$

Note that the $\gamma u$ term added to $L_1 u$ only shifted the eigenvalues. The result:

**For the second kind:** For the FIE of the second kind (4.1), the eigenfunctions are the same as $L_1$ and the eigenvalues are shifted by $\gamma$. That is, the eigenvalues of $L$ are

$$\lambda_1, \cdots, \lambda_n \ (\neq \gamma), \quad \lambda^\infty = \gamma$$

with the **same eigenfunctions** $\phi_1, \cdots, \phi_n$ and $\phi_m^\infty$ $(m = 1, 2, \cdots)$.

It follows that $L$ has all non-zero eigenvalues unless $-\gamma$ is an eigenvalue of $L_1$.

Now we solve the FIE

$$Lu = \gamma u + L_1 u = f \tag{4.2}$$

under the assumption that

$$\lambda_1, \cdots, \lambda_n \neq 0. \tag{4.3}$$

If (4.3) holds then the FAT case (A) applies (since $\lambda^\infty$ is non-zero) and there is a unique solution, unlike an FIE of the first kind.

**Direct method (not ideal):** Expand $u$ and $f$ in terms of the eigenfunctions:

$$u = \sum_{j=1}^n c_j\phi_j + \sum_m c_m^\infty \phi_m^\infty, \qquad f = \sum_{j=1}^n f_j\phi_j + \sum_m f_m^\infty \phi_m^\infty.$$

Now plug into the FIE (4.2) to get

$$Lu = L\left(\sum_{j=1}^n c_j\phi_j + \sum_m c_m^\infty \phi_m^\infty\right) = \sum_{j=1}^n \lambda_j c_j\phi_j + \lambda^\infty \sum_m c_m^\infty \phi_m^\infty.$$

Equate this to $f$ (note that now $\lambda^\infty = \gamma \neq 0$) to get the coefficients:

$$\sum_{j=1}^n \lambda_j c_j\phi_j + \lambda^\infty \sum_m c_m^\infty \phi_m^\infty = f \implies \cdots$$

This can be continued, but there is a better shortcut, inspired by the FIE-1 case. The trick is that the 'infinite part' (with $\sum_m(\cdots)$) of the solution is really a multiple of $f$, leaving only the 'finite part', which is a sum only over $n$ terms.

**Practical note:** The direct method leads to trivial equations (from orthogonality), but there are an infinite number of them. Moreover, the $\sum_m$ part is best avoided if we can get away with a solution containing only a finite sum!

4.1. **Shortcut (undetermined coefficients):** Assume a solution of the form

$$u = \sum_{j=1}^{n} b_j \alpha_j(x) + \frac{1}{p} f(x). \tag{4.4}$$

Since the kernel is separable, $L_1 g$ for any function $g(t)$ is a linear combination of $\alpha_j(x)$'s:

$$L_1 g = \int_a^b k(x,t) g(t)\,dt = \int_a^b \sum_j \alpha_j(x)\beta_j(t) g(t)\,dt = \sum_j \left( \int_a^b \beta_j(t) g(t)\,dt \right) \alpha_j(x). \tag{4.5}$$

Now plug the guess (4.4) into $Lu$ and keep the $f$ part of $\gamma u$ separate from the rest. All the other terms, via rule (4.5), will give various linear combinations of $\alpha_j$'s. To be precise,

$$L_1 \alpha_i = \sum_j c_{ij} \alpha_j(x), \qquad L_1 f = \sum_j d_j \alpha_j(x).$$

Now plug in and collect all the $\alpha_j$ terms and the leftover $f$ part:

$$\begin{aligned} Lu &= \gamma u + L_1 u \\ &= \sum_j \gamma b_j \alpha_j(x) + \frac{\gamma}{p} f + \sum_i b_i (L_1 \alpha_i) + \frac{1}{p} L_1 f \\ &= \sum_j \gamma b_j \alpha_j(x) + \frac{\gamma}{p} f + \sum_j \left( \sum_i c_{ij} b_j + d_j \right) \alpha_j(x) \\ &= \underbrace{\sum_j \gamma b_j \alpha_j(x) + \sum_j \left( \sum_i c_{ij} b_j + d_j \right) \alpha_j(x)}_{=0} + \underbrace{\frac{\gamma}{p} f}_{f} \end{aligned}$$

That is, the $\sum(\cdots)\alpha_j$ part must cancel, leaving just $f$, so

$$p = \gamma,$$
$$b_j = \sum_i c_{ij} b_j + d_j \text{ for } j = 1, 2, \cdots, n$$

which is linear system for $\mathbf{b} = (b_1, \cdots, b_n)$ of the form $\mathbf{b} = C\mathbf{b} + \mathbf{d}$. Now recall that under the assumptions made at the start, the FAT guarantees a unique solution, so we are done (the guess is a solution, so it is the unique solution).

4.2. **Example:** again, return to the example. Now let

$$L_1 u = \int_0^1 k(x,t) u(t)\,dt, \quad Lu = \frac{1}{2} u + L_1 u.$$

The eigenvalues of $L$ are $1/2 + 1/2$ and $1/2 - 1/6$, with the same eigenfunctions as $L_1$ computer earlier. Consider

$$\frac{1}{2} u + L_1 u = e^x.$$

Look for a solution

$$u = b_1 \alpha_1(x) + b_2 \alpha_2(x) + \frac{1}{p} e^x.$$

After following the process[2] (plug everything in), $p = 2$ and

$$u = (72 - 30e)x + (55e - 140)x^2 + 2e^x.$$

This is the unique solution! Note the $2e^x$ deals neatly with the RHS; no infinite sum required. The downside is that the algebra for $b_1$ and $b_2$ is messy, but it is a small linear system, and a computer algebra package has no trouble with it.

## 5. What if the kernel is non-separable?

For a **non-separable kernel** such as

$$k(x,t) = \sum_{j=1}^{\infty} \alpha_j(x)\beta_j(t) \tag{5.1}$$

the theory is different. More conditions are required to ensure that the eigenvalue problem behaves well. If the kernel is 'nice', there is an infinite set of eigenvalues with structure similar to Sturm-Liouville theory. As for Sturm-Liouville operators the general idea[3] is that

$$L \text{ self-adjoint } + (L) \text{ is 'nice'} \implies \text{good properties for eigenfunctions/values.}$$

For integral equations, **Hilbert-Schmidt theory** provides the answer for what is required to be 'nice' and what 'good properties' result. The **Hilbert-Schmidt condition** is

$$\int_a^b \int_a^b k(x,t)^2 \, dx \, dt < \infty \tag{5.2}$$

which guarantees that the operator $\int_a^b k(x,t)f(t)\,dt$ is 'nice' (precisely, a **compact operator** on $L^2[a,b]$). Hilbert-Schmidt theory then says that if

$$Lu = \int_a^b u(t)k(x,t)\,dt$$

satisfies the conditions

1) $L$ is self-adjoint (in $L^2[a,b]$)
2) $k(x,t)$ is not separable (although the separable case is similar)
3) The Hilbert-Schmidt condition (5.2) holds ( $\implies$ $L$ is 'compact')

then the eigenvalue problem has nice properties:

- The eigenvalues $\lambda$ are real
- The eigenvalues are a discrete sequence, each with **finite** multiplicity:
$$|\lambda_1| \le |\lambda_2| \le \cdots$$
- The eigenfunctions $\{\phi_j\}$ are an orthogonal basis for $L^2[a,b]$ ($f = \sum c_j\phi_j$ if $f \in L^2[a,b]$)
- The Fredholm Alternative and eigenfunction expansions for $Lu = f$ apply (in the same way that we used them for BVPs with Sturm-Liouville theory)

For further reading, see the references at the start of the notes. We will study non-separable kernels in a different way when addressing the Fourier and Laplace transforms.

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[2]The computations are messy and not included here.

[3]Both the theory for integral operators and Sturm-Liouville theory are part of a more general theory; the structure comes from the theory of **compact operators** on a Hilbert space (the 'spectral theorem for compact operators').