1. Boundary value problems: introduction

A boundary value problem (BVP) is a differential equation (DE) with constraints specified at more than one boundary. As a simple example, consider

$$\frac{d^2u}{dx^2} = 1, \quad u(0) = 1, \quad u(1) = 2. \quad (1)$$

This is fundamentally different from an ‘initial value problem’ (IVP), e.g.

$$\frac{d^2u}{dx^2} = 1 \quad u(0) = 1, \quad u'(0) = 1$$

where all the ‘boundary data’ is at one point ($x = 0$ here).

For the BVP (1), the solution to the ODE (by integrating twice) is

$$u = \frac{1}{2}x^2 + ax + b$$

and applying the boundary conditions we get

$$u(x) = 1 + (x^2 + x)/2.$$
However, this ‘direct’ approach will not get us far (most BVPs cannot just be integrated like this). For myriad reasons, we will need better methods. To develop them, we will recast the problem in terms of linear algebra. The first step is to introduce the right language.

**Definitions:** Recall that a **linear operator** \( L \) on a vector space \( V \) is a function \( V \to V \) that is **linear**, i.e. 
\[
L(c_1u_1 + c_2u_2) = c_1Lu_1 + c_2Lu_2 \text{ for all scalars } c_1, c_2 \text{ and } u_1, u_2 \in V.
\]

Here we care about linear operators \( L \) acting on functions \( u : [a, b] \to \mathbb{R} \). The **domain** \([a, b]\) of the **functions** is essential. However, the space \( V \) on which the operator is defined (e.g. \( L^2 \) functions on \([a, b]\)) is often not needed (we will be precise where it is necessary).

A **linear differential operator** involves derivatives of the input function, such as 
\[
Lu = x^2 \frac{d^2u}{dx^2} + x \frac{du}{dx} + 2u
\]

A **boundary value problem** has three parts:

- A **domain** e.g. \([a, b]\) (possibly infinite)
- A **differential equation** (DE) 
  \[
  Lu = f, \quad x \in (a, b)
  \]
  for a function \( u \) defined on the domain (the function to solve for) and some linear differential operator \( L \) (just ‘linear operator’ for short).

- **Boundary conditions** (BCs) that are some relations for \( u \) and its derivatives at the boundaries of the domain (see definitions below).

The DE is **homogeneous** if \( f = 0 \) (so \( Lu = 0 \)) and **inhomogeneous** otherwise.

**Definitions (types of BCs)** For a BVP in \([a, b]\), the a boundary condition will have the form 
\[
Bu = c
\]

where \( B \) is linear (see below) and involves \( u \) and its derivatives at \( x = a \) or \( x = b \).

There are several important types of boundary conditions to note:

- **Dirichlet:** \( u(a) = C \) (or \( u(a) = 0 \))
- **Neumann:** \( u_x(a) = C \) (or \( u_x(a) = 0 \))
- **Robin:** \( \alpha u(a) + \beta u_x(a) = C \)
- **Periodic:** \( u(a) = u(b), u'(a) = u'(b), \cdots \)

The BCs are called **separated** if each equation involves only \( a \) or \( b \) (not both) and **mixed** otherwise. E.g. Dirichlet at \( a \) and Neumann at \( b \) is separated; periodic is mixed.

Note that all the BCs here are ‘linear’ in that they have the form \( Bu = c \) where the ‘boundary function’ \( B \) satisfies \( B(c_1u_1 + c_2u_2) = c_1Bu_1 + c_2Bu_2 \).

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1The name is French, so it is pronounced a bit like ‘row-beh’. They are also know as BCs of ‘the third kind’, with Dirichlet/Neumann being the first and second.
The example (1), in the notation outlined above, has
\[ L = d^2u/dx^2, \quad f(x) = 1, \quad B_1u = u(0), \quad B_2u = u(1). \]
For convenience, we denote by \( Bu \) the ‘boundary operator’ that combines all the BCs, e.g.
\[ u(0) = 0, \quad u(1) = 1 \implies Bu = (u(0), u(1)). \]
The point of this is to write statements like ‘\( u \) satisfies the boundary conditions’ as \( Bu = 0 \).

**Notation (BCs in physics):** The main types of BCs have many names:
- Dirichlet: ‘BCs of the first kind’
- Neumann: ‘BCs of the second kind’ or ‘flux BCs’ (when it represents a specified flux through the boundary)
- Robin: ‘BCs of the third kind’ or ‘radiation BCs’

The names Dirichlet/Neumann are by far the most common (at least in mathematics), but ‘flux’ is often used for Neumann when it makes physical sense.

## 2. ADJOINT OPERATORS

A linear operator for a BVP has associated boundary conditions. Because the BCs are so important, we often think of the two together as one entity. To be precise, define:
- **Formal operator:** The operator \( L \) itself, acting on any function defined on the interval \([a, b] \). This is usually what is called the ‘operator’.
- **Complete operator:** The operator \( L \) along with the boundary conditions, acting on functions in \([a, b] \) that satisfy the boundary conditions.

We’ll use the latter to indicate that \( L \) has specified BCs.

The **complete adjoint operator** is an operator \( L^* \) along with **adjoint boundary conditions** \( B^* \) such that
\[ \langle Lu, v \rangle = \langle u, L^*v \rangle \text{ for all } u \text{ s.t. } Bu = 0 \text{ and } v \text{ s.t. } B^*v = 0 \]
The **formal adjoint** is just \( L^* \), which is the operator such that
\[ \langle Lu, v \rangle = \langle u, L^*v \rangle + \text{boundary terms} \]
which will make more sense with examples below. We say:
- \( L \) is **formally self-adjoint** if \( L = L^* \) (roughly, self-adjoint ignoring BCs)
- **self-adjoint** if the formal operators and BCs are equal \( (L = L^* \text{ and } B = B^*) \)

**Important note:** The BCs and adjoint BCs are always **homogeneous**, i.e. in the form \( Bu = 0 \) and not \( Bu = c \); there is no hope of having an adjoint at all in the latter case.

As an example (notation: \( u_x = du/dx \)), we compute the (complete) adjoint operator for
\[ Lu = u_{xx} + p(x)u_x \]
in the domain \([0, 1] \) with boundary conditions
\[ 2u(0) - u'(0) = 0 \]
\[ u'(1) = 0. \]
The trick to use integration by parts to move all derivatives from \( u \) onto \( v \) and get
\[
\langle Lu, v \rangle = \text{‘BC terms’} + \int_0^1 u(\cdot \cdot \cdot) \, dx.
\]
The integral is \( \langle u, L^*v \rangle \) for some operator \( L^* \) that can be identified. Setting the BC terms to zero for all \( u \) will give the adjoint boundary conditions for \( v \).

The details for the example go as follows. First, integrate by parts repeatedly:
\[
\langle Lu, v \rangle = \int_0^1 u_{xx}v \, dx + \int_0^1 p(x)u_xv \, dx
\]
\[
= u_xv\big|_0^1 - \int_0^1 u_xv_x \, dx + puv\big|_0^1 - \int_0^1 u(pv)_x \, dx
\]
\[
= (u_xv - uv_x + puv)\big|_0^1 + \int_0^1 u(v_{xx} - (pv)_x) \, dx
\]
\[
= \text{‘BC terms’} + \langle u, L^*v \rangle
\]
For the \( \langle u, L^*v \rangle \) part, the integral is \( \int_0^1 uL^*v \, dx \) for
\[
L^*v = v_{xx} - (pv)_x
\]
which is the formal adjoint. Now the ‘BC terms’ must vanish given adjoint BCs to be found. Since the BCs are separated we may look at \( x = 0 \) and \( x = 1 \) part of the ‘BC terms’ separately. Use the BCs for \( u \) to simplify (replace \( u'(0) \) with \( 2u(0) \)):
\[
(\text{BC term, } x = 0) = u_x(0)v(0) - u(0)v_x(0) + p(0)u(0)v(0)
\]
\[
= u(0)(2v(0) - v_x(0) + p(0)v(0)).
\]
This must vanish for all values of \( u(0) \) (since this is not specified at \( x = 0 \)) so
\[
(2 + p(0))v(0) - v_x(0) = 0.
\]
Now do the same for the other boundary term:
\[
(\text{BC term, } x = 1) = u_x(1)v(1) - u(1)v_x(1) + p(1)u(1)v(1)
\]
\[
= u(1)(-v_x(1) + p(1)v(1))
\]
which must vanish for all values of \( u(1) \) so
\[
p(1)v(1) - v_x(1) = 0.
\]
To summarize, the original and adjoint operators are
\[
Lu = u_{xx} + p(x)u_x
\]
\[
L^*v = v_{xx} - (pv)_x
\]
and the original and adjoint boundary conditions are
\[
2u(0) - u'(0) = 0 \quad (BC)
\]
\[
u'(1) = 0.
\]
\[
(2 + p(0))v(0) - v_x(0) = 0 \quad (BC^*)
\]
\[
p(1)v(1) - v_x(1) = 0.
\]
This ensures that \( \langle Lu, v \rangle = \langle u, L^*v \rangle \) for all \( u \) satisfying (\( BC \)) and \( v \) satisfying (\( BC^* \)).
Example (self-adjoint operator): We show that the complete operator
\[ Lu = -(pu)_x + qu, \quad u(0) = u(1) = 0 \]
is self-adjoint (where \( p(x) \) and \( q(x) \) are functions). Integrate by parts twice:
\[
\langle Lu, v \rangle = -\int_0^1 (pu)_x v \, dx + \int_0^1 quv \, dx
\]
\[
= -pu_x v \bigg|_0^1 + \int_0^1 pu_x v_x \, dx + \int_0^1 quv \, dx
\]
\[
= -pu_x v \bigg|_0^1 + puv \bigg|_0^1 - \int_0^1 u(pv_x)_x \, dx + \int_0^1 quv \, dx.
\]
The integral is \( \langle u, L^* v \rangle \) for the operator
\[ L^* v = -(pv_x)_x + qv \]
so \( L = L^* \) (formally self-adjoint). For the BCs the second term cancels, leaving
\[
p(0)u_x(0)v(0) = 0 \implies v(0) = 0
\]
\[
p(1)u_x(1)v(1) = 0 \implies v(1) = 0
\]
since the boundary terms must vanish for all values of \( u_x(0) \) and \( u_x(1) \) (both arbitrary). Thus the adjoint BCs are \( v(0) = v(1) = 0 \), the same as the BCs for \( L \), so \( L \) is self-adjoint.

3. Eigenvalue problems: three familiar examples

An operator \( L \) in \([a, b]\) with homogeneous boundary conditions \( Bu = 0 \) has an associated eigenvalue problem to find an eigenfunction \( \phi \) in \([a, b]\) and an eigenvalue \( \lambda \) such that
\[ L\phi = \lambda\phi, \quad B\phi = 0. \] (3)

Procedure for eigenvalue problems: The general procedure for solving the eigenvalue problem (3) is

a) In each range of \( \lambda \) where the DE has a certain form, find the general solution
\[ \phi = c_1\phi_1 + \cdots + c_n\phi_n \]
where \( n \) is the order of the DE, using standard ODE solving techniques.

b) Use the boundary conditions \( B\phi = 0 \) to get \( \approx n \) equations for the \( c \)'s (plus any other constraints relevant to the problem from elsewhere)

c) Find all \( \lambda \) such that there are non-trivial solutions (\( c \)'s not all zero) and identify the eigenfunction (basis for each set of solutions to \( L\phi = \lambda\phi \))

There are three standard examples. Consider the operator
\[ Lu = -\frac{d^2u}{dx^2} \]
in \([0, \pi]\) with three different boundary conditions. The eigenfunctions should look familiar.
Notation: The minus sign is partly for historical reasons. It makes the eigenvalues all positive (rather than all negative), which is convenient.

3.1. Dirichlet BCs. The eigenvalue problem for \( \phi(x) \) is

\[
-\phi'' = \lambda \phi, \quad x \in (0, \pi), \quad \phi(0) = 0, \quad \phi(\pi) = 0.
\]

To solve it, find the general solution to the DE, then use the boundary conditions and look for values of \( \lambda \) such that a non-trivial solution exists.

Case 1: \( \lambda < 0 \). We will show here that no solutions exist. The characteristic polynomial is

\[
r^2 + \lambda = 0.
\]

which has roots \( \pm \mu \) where \( \mu = \sqrt{-\lambda} \). Then the general solution is

\[
\phi = c_1 e^{\mu x} + c_2 e^{-\mu x}.
\]

Apply the boundary conditions to get a system for coefficients \( c_1, c_2 \):

\[
0 = c_1 + c_2, \quad 0 = c_1 e^{\pi \mu} + c_2 e^{-\pi \mu}.
\]

This system, in matrix form, is

\[
\begin{pmatrix}
1 & 1 \\
\text{e}^{-\pi \mu} & \text{e}^{\pi \mu}
\end{pmatrix}
\begin{pmatrix}
c_1 \\
c_2
\end{pmatrix}
= \begin{pmatrix}
0 \\
0
\end{pmatrix}.
\]

It has a non-trivial solution if and only if the determinant is zero. But

\[
\text{det}(\cdots) = e^{-\pi \mu} - e^{\pi \mu} = 2 \sinh(\pi \mu) > 0,
\]

so there is only the trivial solution (i.e. \( c_1 = c_2 = 0 \) is the only solution) for all \( \mu \). We conclude there are no negative eigenvalues.

Case 2: \( \lambda = 0 \). The general solution is \( \phi = c_1 x + c_2 \). Applying the boundary conditions, we need \( b = c_2 \) and \( c_1 \pi + c_2 = 0 \) which forces \( c_1 = c_2 = 0 \). So again, no eigenvalues in this case.

Case 3: \( \lambda > 0 \). The general solution is

\[
\phi = c_1 \sin(\sqrt{\lambda} x) + c_2 \cos(\sqrt{\lambda} x).
\]

From the boundary conditions,

\[
c_2 \cos 0 = 0 \quad c_1 \sin(\sqrt{\lambda} \pi) + c_2 \cos(\sqrt{\lambda} \pi) = 0
\]

so \( c_2 = 0 \). To have a non-trivial solution (\( c_1 \neq 0 \)) we need

\[
\sin(\sqrt{\lambda} \pi) = 0.
\]

This has a non-trivial solution when

\[
\sqrt{\lambda} \pi = n \pi, \quad n = 1, 2, \ldots
\]
i.e. for eigenvalues $\lambda_n = n^2$ (note that $\lambda$ had to be positive to get here, so the $-\sqrt{\lambda}$ root is discarded!). Plugging back into the general solution (recall that $c_2 = 0$) we obtain the corresponding eigenfunctions

$$\phi_n = \sin nx.$$  

**Summary:** Collecting the results of the three cases, we find that only the $\lambda > 0$ case yields eigenvalues. The total set of eigenvalues/functions is

$$\lambda_n = n^2, \phi_n = \sin nx, \quad n = 1, 2, 3, \cdots,$$

which is exactly the basis from the Fourier sine series.

3.2. **Neumann BCs.** The eigenvalue problem is

$$-\phi'' = \lambda \phi, \quad x \in (0, \pi), \quad \phi'(0) = 0, \quad \phi'(\pi) = 0.$$  

**Case 1:** $\lambda < 0$. Again let $\mu = \sqrt{-\lambda}$. The general solution is the same as before, and the boundary conditions require (check this!)

$$0 = \sqrt{\mu}(c_1 - c_2), \quad 0 = \sqrt{\mu} \left(c_1 e^{\pi \mu} - c_2 e^{-\pi \mu}\right).$$

Again write

$$\begin{pmatrix} 1 & -1 \\ e^{\pi \mu} & -e^{-\pi \mu} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$  

The determinant is $-2\mu \sinh(\pi \mu) < 0$. Note that $e^{\pi \mu} > 1$ and $e^{-\pi \mu} < 1$ since $\mu > 0$. Thus there are still no solutions.

**Case 2:** $\lambda = 0$. The general solution is

$$\phi = c_1 x + c_2.$$  

The boundary conditions require only that $c_1 = 0$, so $\phi = 1$ is an eigenfunction for $\lambda = 0$.

**Case 3:** $\lambda > 0$. The general solution is the same as before; the boundary conditions require

$$c_1 = 0, \quad -\sqrt{\lambda}c_2 \sin(\sqrt{\lambda} \pi) = 0.$$  

Thus we need $\sqrt{\lambda} \pi = n\pi$, so the eigenvalues are

$$\lambda_n = n^2, \quad n = 1, 2, \cdots$$

with corresponding eigenfunctions

$$\phi_n = \cos nx.$$  

So in summary, the eigenfunctions/values are

$$\lambda_n = n^2, \quad \phi_n = \cos nx, \quad n = 0, 1, 2, \cdots,$$

which are the basis functions for the Fourier cosine series. Note that $n = 0$ is included here (from Case 2), which is absent from the Dirichlet version.
3.3. Periodic BCs. Take the domain to be $[0, 2\pi]$ for reasons to be clear. The problem is

$$-\phi'' = \lambda \phi \quad x \in (0, 2\pi), \quad \phi(0) = \phi(2\pi), \quad \phi'(0) = \phi'(2\pi).$$

First, if $\lambda = 0$ then

$$\phi = c_1 x + c_2.$$

The boundary conditions are satisfied if

$$c_2 = c_1 (2\pi) + c_2 \implies c_1 = 0, \quad c_1 = c_1$$

so $\phi = \text{const}$ is an eigenfunction for $\lambda = 0$.

If $\lambda < 0$ the solution is, with $\mu = \sqrt{-\lambda}$,

$$\phi = c_1 e^{\mu x} + c_2 e^{-\mu x}.$$

From the boundary conditions,

$$c_1 + c_2 = c_1 e^{2\pi \mu} + c_2 e^{-2\pi \mu}, \quad \mu c_1 + \mu c_2 = c_1 \mu e^{2\pi \mu} - c_2 \mu e^{-2\pi \mu}.$$

This becomes the system

$$\begin{bmatrix}
1 - e^{2\pi \mu} & 1 - e^{-2\pi \mu} \\
1 - e^{2\pi \mu} & 1 + e^{-2\pi \mu}
\end{bmatrix}
\begin{bmatrix}
c_1 \\
c_2
\end{bmatrix}
= \begin{bmatrix}
0 \\
0
\end{bmatrix}.$$

There is a solution if and only if the determinant is zero, so

$$0 = (1 - e^{2\pi \mu})(1 + e^{-2\pi \mu}) - (1 - e^{2\pi \mu})(1 - e^{-2\pi \mu}) = 2(1 - e^{2\pi \mu}) e^{-2\pi \mu}.$$

Thus $e^{2\pi \mu} = 1 \implies \mu = 0$ but $\mu$ has to be positive (since $\lambda < 0$) so there are no solutions.

If $\lambda > 0$ then, setting $\mu = \sqrt{\lambda}$, the solution is

$$\phi = c_1 \cos \mu x + c_2 \sin \mu x.$$

From the boundary conditions,

$$c_1 = c_1 \cos 2\pi \mu + c_2 \sin 2\pi \mu, \quad c_2 \mu = -\mu c_1 \sin 2\pi \mu + \mu c_2 \cos 2\pi \mu$$

As a system:

$$\begin{bmatrix}
1 - \cos(2\pi \mu) & -\sin(2\pi \mu) \\
\sin 2\pi \mu & 1 - \cos 2\pi \mu
\end{bmatrix}
\begin{bmatrix}
c_1 \\
c_2
\end{bmatrix}
= \begin{bmatrix}
0 \\
0
\end{bmatrix}.$$  \hspace{1cm} (4)

Set the determinant to zero:

$$0 = (1 - \cos(2\pi \mu))^2 + \sin^2(2\pi \mu) = 2 - 2 \cos(2\pi \mu).$$

Thus there is a solution for

$$\cos(2\pi \mu) = 1$$

which occurs for the values $\mu_n = n$ for $n = 1, 2, \ldots$.

Notice that for such values, (4) becomes trivial and any $c_1$ and $c_2$ will work, so

$$\phi = c_1 \cos nx + c_2 \sin nx$$

is a solution for all $n \geq 1$ with eigenvalue $\lambda_n = n^2$. This means we have eigenfunctions

$$\phi_n = \cos nx, \quad \psi_n = \sin nx \quad \text{with} \quad \lambda_n = n^2 \quad \text{for} \quad n = 1, 2, \ldots$$

where $\phi_n$ and $\psi_n$ are both eigenfunctions for $\lambda_n$. This is, of course, the basis for the full Fourier series (computed on $[0, 2\pi]$ rather than $[-\pi, \pi]$, but it is the same up to this translation).
3.4. **Cauchy-Euler equations.** Other than constant coefficient linear equations, there is one other useful type of ODE we can solve exactly: an **Euler equation** (or ‘Cauchy-Euler’ or ‘equidimensional’) ODE, which has the form

$$Lu := ax^2 \frac{d^2 u}{dx^2} + bx \frac{du}{dx} + cu = 0. \quad (5)$$

Assume that $x > 0$ for simplicity. We see two basis solutions that span the solution set. To find them, first look for solutions of the form $u = x^r$. We get

$$L[x^r] = (ar(r-1) + br + c)x^r$$

so it follows that

$x^r$ is a soln $\iff p(r) = ar(r-1) + br + c = 0$.

If $p$ has two real roots $r_1, r_2$ then the general solution is the span of the two solutions,

$$u(t) = c_1 x^{r_1} + c_2 x^{r_2}.$$  

If $p$ has complex roots $r = s \pm \omega i$ then write

$$x^r = e^{r \log x} = x^s e^{\omega i \log x} = x^s (\cos \omega \log x + i \sin \omega \log x).$$

Taking real/imaginary parts gives the two basis solutions so the general solution is

$$x = x^s (c_1 \cos \omega \log x + c_2 \sin \omega \log x).$$

Finally, if $p$ has a repeated root $r$ then the general solution is

$$x = x^r (c_1 + c_2 \log x).$$

That is, multiply the first solution by $\log x$ to get the second.

The ‘eigenvalue problems’ are solved the same way as the previous examples, now using this general solution. The procedure also works for higher-order ‘equidimensional’ equations, although the repeated roots case may need to be generalized if there is a triple root.