Topics covered

• Complex functions
  ◦ Complex differentiability; analytic functions
  ◦ Cauchy-Riemann equations (connection to harmonic functions)
  ◦ Multiple-valued functions and branch cuts
• Preparing for contour integrals
  ◦ Poles and meromorphic functions
  ◦ Cauchy integral formula, Laurent series
  ◦ Contour integrals; contour deformation (in general)
  ◦ Residue theorem, calculating residues
  ◦ Trick for simple poles

Introduction

The main goal here is to introduce the fundamentals of complex analysis required to work with contour integrals that arise in the Fourier and Laplace transforms and the nice properties of complex functions that inform the solutions. Our focus will be on these transform methods for solving PDEs, but it will be necessary to first learn the methods for computing contour integrals in general.

As such, the treatment here is minimal and is by no means a thorough look at this rather deep subject. We will leave out some topics in favor of having more time to focus on the intended applications.

Notation: The following notation will be used (refer back as needed):

• \( i = \sqrt{-1} \) (the imaginary unit) and \( z = x + iy \) is a complex number with real/imaginary parts \( x \) and \( y \). A point \( z \) in the complex plane \( \mathbb{C} \) is associated with \( (x, y) \) in \( \mathbb{R}^2 \).
• A complex function \( f(z) \) with real/imaginary parts \( u \) and \( v \) can be written
  \[
  f(z) = f(x + iy) = u(x, y) + iv(x, y)
  \]
• \( \Gamma \) is a contour in the complex plane, often parameterized by \( z(t) = (x(t), y(t)) \)
• \( \int_{\Gamma} f \, d\zeta \) is a contour integral in the complex plane and \( \oint_{\Gamma} f \, dz \) is a contour integral over a closed contour (or 'closed loop')

1. Review of complex numbers

• Complex numbers: \( \mathbb{C} \) is the space of complex numbers \( z = x + iy \) where \( i = \sqrt{-1} \) is the ‘imaginary unit’ and
  \[
  x = \text{Re}(z) = \text{‘real part’}, \quad y = \text{Im}(z) = \text{‘imaginary part’}.
  \]
Addition is component-wise as in $\mathbb{R}^2$ but $i^2 = -1$ so

$$z_1 z_2 = (x_1 + y_1 i)(x_2 + y_2 i) = x_1 x_2 - y_1 y_2 + (y_1 x_2 + x_1 y_2)i.$$ 

While tempting to compare to the plane $\mathbb{R}^2$, the ‘complex plane’ has much more structure due to the multiplication of complex numbers.

The \textbf{conjugate} (with an overbar) of $z = x + iy$ and the \textbf{modulus} $|z|$ (‘magnitude’) are

$$\overline{z} = x - yi, \quad |z| = \sqrt{z \overline{z}} = \sqrt{x^2 + y^2}.$$ 

The conjugate distributes over sums/products ($vw = \overline{vw}$ etc.) and

$$\text{Re}(z) = \frac{1}{2}(z + \overline{z}), \quad \text{Im}(z) = \frac{1}{2}(z - \overline{z}).$$

\begin{itemize}
  \item \textbf{Polar form:} A complex number $z$ can be written in rectangular or polar coordinates:

  $$z = x + iy \text{ or } z = r \cos \theta + ir \sin \theta, \quad r^2 = x^2 + y^2, \quad \theta = \arg z = \tan^{-1}(y/x)$$

  The values $r$ and $\theta$ are the \textbf{modulus} and \textbf{argument}, respectively.

\end{itemize}

\textbf{Euler’s formula and $n$-th roots:} A critical result is \textbf{Euler’s formula},

$$\cos \theta + i \sin \theta = e^{i \theta} \quad \text{(so } z = re^{i \theta} \text{ in polar form)}$$

In particular, note that this implies that $e^{2\pi i} = 1$ and

$$z^n = 1, \quad n \geq 1 \text{ an integer } \implies z = e^{2\pi ik/n}, \quad k = 0, \ldots, n - 1$$

More generally, solving $z^n = c$ is done in polar form:

$$z^n = re^{i \alpha} \implies z = r^{1/n} e^{i \alpha/n} e^{2\pi ik/n}, \quad k = 0, \ldots, n - 1$$

Other arithmetic is also easy in this form e.g. $e^{i \theta} = e^{-i \theta}$ and $r_1 e^{i \theta_1} \cdot r_2 e^{i \theta_2} = r_1 r_2 e^{i(\theta_1 + \theta_2)}$.

\begin{itemize}
  \item \textbf{Functions:} A general complex function is a function of two variables, typically written as $f(z, \overline{z})$ or a function of $x, y$ for the input $z = x + iy$. It can be written in terms of real/imaginary components:

  $$f(z, \overline{z}) = u(x, y) + iv(x, y)$$

  where $u, v$ are real functions from the plane to $\mathbb{R}$. Most real functions $f(x)$ have ‘complexified’ versions $f(z)$ (just a function of $z$) e.g.

  $$f(x) = x^2 \implies f(z) = (x + iy)^2 = x^2 - y^2 + i2xy.$$ 

\end{itemize}
2. Multi-valued functions, branch cuts

A complex function can be multi-valued due to the formula
\[ e^{2\pi ik} = 1 \text{ for all integers } k. \]

Most functions are still single-valued despite this, e.g.
\[ f(z) = z^2 \implies f(z e^{2\pi ik}) = z^2 e^{4\pi ik} = z^2. \]

However, multi-valued functions are also common. For example, consider
\[ f(z) = z^{1/2}. \]

Evaluating in polar coordinates with \( z = re^{i\theta} \),
\[ f(re^{i\theta}) = f(r e^{i\theta} \cdot 1) = f(r e^{i\theta + 2\pi k}) = r^{1/2} e^{i\theta/2} e^{i\pi k}, \quad k = 0 \text{ or } 1. \]

That is, there are two versions of \( f \) that can be defined:
\[
\begin{align*}
  f(z) &= |z|^{1/2} e^{i\theta/2} & (k = 0) \\
  f(z) &= -|z|^{1/2} e^{i\theta/2} & (k = 1)
\end{align*}
\]

The two definitions are the branches of \( f \). We must choose one branch to have a single-valued function. The \( k = 0 \) version is called the principal branch of the function. Regardless of which branch is chosen, there is a jump across some line (segment) in the complex plane:

The location of this line depends on the choice of argument. If we take \( \theta \in [-\pi, \pi] \) then there is a jump at \( \theta = \pi \) since
\[ f(e^{i\theta}) \to i \text{ as } \theta \text{ increases to } \pi, \quad f(e^{i\theta}) \to -i \text{ as } \theta \text{ decreases to } -\pi \]
so there is a jump at \( \theta = \pi \). The problem is that the argument \( \theta = \arg z \) is periodic, but \( e^{i\arg z/2} \) has different values for \( \theta \)'s separated by \( 2\pi \). We could also choose \( \theta \in [0, 2\pi] \), in which case the jump is across \( \theta = 0 \) instead.

The segment across which \( f \) is discontinuous is the branch cut and the endpoints are the branch points. In this case, we have one branch point at 0 and another ‘at \( \infty \)’ (at \( -\infty + 0i \) with \( \theta \in [-\pi, \pi] \) and \( \infty + 0i \) for \( \theta \in [0, 2\pi] \).

- **Logarithms**: An important example is the natural logarithm:
\[
\ln(z) = \ln(r e^{i\theta} e^{2\pi ik}) = \ln(r) + i\theta + 2\pi ik \\
= \ln|f(z)| + i \arg(z) + 2\pi ik
\]
with **infinite number of branches** (one for each integer $k$). The principal branch is

$$\ln(z) = \ln(r) + i\theta.$$  

This function has a branch cut where $\arg(z)$ jumps and a branch point at the origin, and discontinuous across this jump (by a value of $2\pi i$).

- Typically, the branch cut occurs where $\arg(z)$ jumps, and the range of $\theta$ determines the placement of the branch cut. For more complicated functions (to be discussed later), branch cuts can be segments or centered at branch points other than the origin.

- **Roots in general:** Using the natural logarithm we may determine branch cuts for ‘$n$-th roots’ and other powers using the definition

$$z^p = e^{p \ln z}.$$  

For example, consider $(z - z_1)^{1/m}$. Using the above and the formula for $\ln z$,

$$(z - z_1)^{1/m} = \exp\left( \frac{1}{m} \ln(z - z_1) \right) = |z - z_1|^{1/m} \exp\left( \frac{2\pi ik}{m} \arg(z - z_1) \right)$$

With $z - z_1 = re^{i\alpha}$, this reads

$$(z - z_1)^{1/m} = r^{1/m} \exp(2\pi ik\alpha/m).$$

Each distinct value of $\exp(2\pi ik\alpha/m)$ yields a branch; there may be finite or infinitely many depending on the number of distinct values.

**Example (branch cut):** Suppose we wish to define a branch of $z^4$ so that 

$$f(z) = z^4, \quad f(1) = i, \quad f \text{ is continuous along the real axis (except 0).}$$

There are four choices for $z^4$, from the calculation

$$f(ze^{2\pi ik}) = |z|^{1/4} e^{i\theta/4} e^{\pi k/2}.$$  

The one with $f(1) = i$ is (plug in $z = e^{i\theta}$)

$$z^4 = |z|^{1/4} e^{i\theta/4} e^{i\pi/2}. $$

The branch cut occurs at the endpoint of the range for $\theta = \arg z$. We needed continuity at $\theta = 0$ and $\theta = \pi$ so the branch cut must be placed somewhere else e.g. by taking $[-\pi/2, 3\pi/2]$ (any range including 0, $\pi$ inside). This puts the branch cut at $-\pi/2$.  

![Diagram showing branch cut](image-url)
3. Analytic functions

A complex function \( f(z, \overline{z}) \) has the form
\[
f(z, \overline{z}) = u(x, y) + iv(x, y), \quad x = \frac{z + \overline{z}}{2}, \quad y = \frac{z - \overline{z}}{2}.
\]

(3.1)
The function is called analytic or holomorphic if it is well-defined and solely a function of \( z \). The ‘brute force’ meaning of this is that if the expressions for \( x, y \) are plugged into (3.1), the \( \overline{z} \)'s cancel, leaving only \( z \) (so \( f = f(z) \)).

This is equivalent to no dependence on \( \overline{z} \), written succinctly as
\[
f = f(z) \iff \frac{\partial f}{\partial \overline{z}} = 0.
\]

(3.2)
The complex derivative of an analytic function is
\[
f'(z) = \lim_{\Delta z \to 0} \frac{f(z + \Delta z) - f(z)}{\Delta z}.
\]

(3.3)
The limit can be taken over \( \Delta z = \Delta x + i\Delta y \) as \( \Delta x, \Delta y \to 0 \) in any way. Thus (3.3) is really two partial derivatives. However, when \( f \) is analytic, (3.3) is well-defined - the limit in any direction \( (\Delta x, \Delta y \to 0 \) by any path) will give the same value \( f'(z) \).

To understand why \( f'(z) \) is well defined, let \( z = x + iy \), let \( f \) be analytic and write
\[
f(z) = u(x, y) + iv(x, y)
\]
We can take the derivative along the ‘real’ or ‘imaginary’ directions:
\[
\Delta z = \Delta x + i\Delta y \to 0 \text{ with } \begin{cases} \Delta x \to 0, \Delta y = 0 & \text{(real)} \\ \Delta x = 0, i\Delta y \to 0 & \text{(imaginary)} \end{cases}
\]
This gives two values for \( f'(z) \), which are
\[
f'(z) = \lim_{\Delta x \to 0} \frac{f(z + \Delta x) - f(z)}{\Delta x} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}
\]
\[
f'(z) = \lim_{i\Delta y \to 0} \frac{f(z + i\Delta y) - f(z)}{i\Delta y} = -i\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}
\]
Equating the two, we obtain the Cauchy-Riemann equations, which also characterize analytic functions. Putting the results in a nice box:

Analytic functions: For \( f = u(x, y) + iv(x, y) \), the following are equivalent:

- \( f \) is well-defined and ‘a function of \( z \) only’ (\( f = f(z) \) or equiv. \( \partial f/\partial \overline{z} = 0 \))
- \( f(z) \) is complex differentiable in the sense that (3.3) exists.
- The real/imaginary parts \( u, v \) satisfy the Cauchy-Riemann equations
\[
\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}.
\]

(3.4)
A function with these properties is called (complex) analytic or holomorphic.
Aside: harmonic functions: Suppose \( f(z) = u(x, y) + iv(x, y) \) and \( u \) and \( v \) are smooth. Letting subscripts denote partial derivatives, we have that
\[
    u_x = v_y, \quad v_x = -u_y.
\]
Now take \( \partial / \partial x \) of the first CR equation and \( \partial / \partial y \) of the second and use the other to simplify:
\[
    u_{xx} = (v_y)_x = (v_x)_y = -u_{yy} \\
    v_{xx} = -(u_y)_x = -(u_x)_y = -v_{xx}
\]
so it follows that \( u, v \) are both harmonic functions (solutions to Laplace’s equation):
\[
    \nabla^2 u = 0, \quad \nabla^2 v = 0.
\]
We can use this fact to solve Laplace’s equation \( \nabla^2 u = 0 \) (beyond the scope of the discussion here) by finding the right \( f(z) \) whose real part satisfies the desired boundary conditions. For example, \( f(z) = z^2 \) is analytic and, with \( z = x + iy \),
\[
    z^2 = x^2 - y^2 + 2ixy \implies u = x^2 - y^2, \quad v = 2xy \text{ solve Laplace’s equation.}
\]

4. Contour Integrals

The starting point is the regular line integral in \( \mathbb{R}^2 \). Analyticity will provide additional structure that we can exploit to evaluate such integrals in \( \mathbb{C} \).

Line integrals (review): Let \( \mathbf{v}(x,y) = (p(x,y), q(x,y)) \) be a vector field in the plane. Let \( \Gamma \) be a path from a point \( A \) to a point \( B \) with parameterization \( \mathbf{x}(t) = (x(t), y(t)) \) (from \( t_0 \) to \( t_1 \)). Then
\[
    \int_{\Gamma} \mathbf{v} \cdot d\mathbf{x} = \int_{t_0}^{t_1} \mathbf{v} \cdot \frac{d\mathbf{x}}{dt} \, dt = \int_{\Gamma} p \, dx + q \, dy = \int_{t_0}^{t_1} p \, dx + q \, dy \, dt.
\]
A vector field \( \mathbf{v} = (p, q) \) is conservative if \( \mathbf{v} = \nabla \phi \) for a ‘potential’ \( \phi(x,y) \). We have:
\[
    \mathbf{v} \text{ is conservative } \iff \int_{\Gamma} \mathbf{v} \cdot d\mathbf{x} \text{ is path-independent } \iff \frac{\partial q}{\partial x} = \frac{\partial p}{\partial y}. \tag{4.1}
\]
Path-independence means the integral depends only on the endpoints of the path:
\[
    \Gamma \text{ from } A \to B \implies \int_{\Gamma} \mathbf{v} \cdot d\mathbf{x} = \phi(B) - \phi(A).
\]
Green’s theorem’s in the plane (the divergence/Stokes’ theorem) state that
\[
    \oint_{\partial \Omega} -q \, dx + p \, dy = \int_{\Omega} \frac{\partial p}{\partial x} + \frac{\partial q}{\partial y} \, dA \\
    \oint_{\partial \Omega} p \, dx + q \, dy = \int_{\Omega} \frac{\partial q}{\partial x} - \frac{\partial p}{\partial y} \, dA
\]
if \( p \) and \( q \) do not blow up in \( \Omega \). Note that the result does not hold otherwise.
4.1. **Contour integrals (analytic functions).** First, some notation:

- Given a path $\Gamma$ from $A$ to $B$, we define the ‘reverse’ path $-\Gamma$ to be the same curve, traversed in the opposite direction. An important rule is that

$$\int_\Gamma f(z) \, dz = - \int_{-\Gamma} f(z) \, dz. \quad (4.2)$$

- A ‘closed contour’ is a path $\Gamma$ from $A$ back to itself. We write $\oint_\Gamma \cdots \, dz$ (with a circle) to denote integrals over closed contours. For a bounded region $\Omega$, the boundary contour is the boundary traversed counter-clockwise.

- This defines the inside of a closed contour $\Gamma$. To check, use the ‘right hand rule’: if the index finger of your right hand points along the path, your thumb points into the region (the inward normal).

In general, for a complex function $f(z, \bar{z}) = u + iv$, the ‘contour integral’ over a path (‘contour’) in the complex plane is a line integral with real and imaginary parts:

$$\int_\Gamma f \, dz = \int_\Gamma (u + iv)(dx + i dy) = \int_\Gamma u \, dx - v \, dy + i \int_\Gamma u \, dx + v \, dy.$$ 

For a general $f$, there is not much more to say. However, when $f$ is analytic, the contour integral has much nicer properties due to the Cauchy-Riemann equations.

**Theorem (path independence):** Let $f(z)$ be analytic in a region $\Omega$ with anti-derivative $F(z) = \int f \, dz$. If $\Gamma$ is a path in $\Omega$ from $A$ to $B$ then

$$\int_\Gamma f(z) \, dz = F(B) - F(A)$$

which depends only on the endpoints and is independent of the path.

**Proof.** To see this, use the theorem for conservative vector fields (4.1). By definition,

$$\int_\Gamma f \, dz = \left( \int_\Gamma u \, dx - v \, dy \right) + i \left( \int_\Gamma v \, dx + u \, dy \right)$$

= line int. of $(u, -v)$ + $i$ (line int. of $(v, u)$)

From the Cauchy-Riemann equations, $v_x = -u_y$ and $u_x = v_y$ so using theorem (4.1),

$$\frac{\partial}{\partial y}(-v) = \frac{\partial}{\partial x}(u) \implies (u, -v) \text{ is conservative}$$
\[ \frac{\partial}{\partial x}(u) = \frac{\partial}{\partial u}(v) \implies (v, u) \text{ is conservative.} \]

It follows that both the real/imaginary parts of \( \int_\Gamma f \, dz \) are path-independent. \( \square \)

Contour integrals can thus be computed without dealing with the path, e.g.

\[ \begin{aligned}
\Gamma &= \text{any path from } 0 \text{ to } 0 + 2i \\
f(z) &= z^2
\end{aligned} \]

\[ \implies I = \int_\Gamma z^2 \, dz = \frac{1}{3} z^3 \bigg|_0^{0+2i} = 8i^3 / 3 = -8i / 3 \]

since \( f(z) = z^2 \) is analytic on all of \( \mathbb{C} \).

Now suppose \( \Gamma \) is a closed contour and \( f \) is analytic inside \( \Gamma \). This curve is the union of a path \( \Gamma_1 \) from \( A \) to \( B \) and a path \( \Gamma_B \) from \( B \) to \( A \) (see sketch). Because \( f \) is analytic, \( f \) has a single anti-derivative \( F(z) \). Thus, if the path independence theorem applies,

\[ \oint_\Gamma f \, dz = \int_{\Gamma_1} f \, dz + \int_{\Gamma_B} f \, dz = ((F(B) - F(A)) + (F(A) - F(B)) = 0. \]  \( (4.3) \)

**Cauchy's theorem:** Let \( \Gamma \) be a closed contour. If \( f(z) \) is analytic inside \( \Gamma \) (no singularities),

\[ \oint_\Gamma f \, dz = 0. \]

Importantly, the result applies only if \( f(z) \) is analytic in the region inside the contour. For example, let \( \Gamma \) be the circle of radius 1, traversed counter-clockwise:

\[ \Gamma = \{ z(t) = e^{it}, \quad t \in [0, 2\pi] \}. \]

Then \( 1/(z - 2) \) is analytic in the region \( |z| \leq 1 \), so

\[ \oint_{\Gamma} \frac{1}{z - 2} \, dz = 0. \]

However, \( 1/z \) is not analytic inside, and

\[ \oint_{\Gamma} \frac{1}{z} \, dz = \int_0^{2\pi} e^{-it} z'(t) \, dt = \int_0^{2\pi} 1 \, dt = 2\pi. \]

The integral \( \Gamma \) is path-independent along any arc: but fails for the full closed contour. The argument in (4.3) does not apply because \( f \) does not have a single anti-derivative in the circle: \( \int 1/z \, dz = \ln z \) is multi-valued. We must place a branch cut somewhere!
4.2. Deformation. A function can fail to be analytic for two reasons: either it diverges at a point, or there is a branch cut (the multi-valued problem). Recall that we must choose a branch of a multi-valued $f(z)$ to make it uniquely defined. To summarize:

**Singularities:** A singularity is a point where a complex function fails to be analytic. There are two main types:

- A ‘pole’ of order $m$ at $z = z_0$, where
  \[ f(z) \approx \frac{C}{(z - z_0)^m} \text{ as } z \to z_0. \]

- A ‘branch cut’, chosen when specifying a branch of $f(z)$ - usually a ray from some point (the branch point) out to $\infty$. The singularity is the entire ray, and $f(z)$ is typically discontinuous across this line.

A function that is analytic except at an isolated set of poles is called **meromorphic**.

Path independence applies only when $f$ is analytic. If this is the case, a contour $\Gamma_A$ can be freely ‘deformed’ into another contour $\Gamma_B$ with the same endpoints and preserve the value:

\[
\int_{\Gamma_A} f \, dz = \int_{\Gamma_B} f \, dz \text{ if } f \text{ is analytic between } \Gamma_A \text{ and } \Gamma_B.
\]

The deformation must avoid any singularity of $f$. To prove this (see below), observe that the contour formed by $\Gamma_A$ and $-\Gamma_B$ is closed and contains no singularities, so

\[
0 = \int_{\Gamma_A} f \, dz + \int_{-\Gamma_B} f \, dz \quad \Rightarrow \quad 0 = \int_{\Gamma_A} f \, dz - \int_{\Gamma_B} f \, dz
\]

by flipping the direction of the second integral.

For closed contours: Similarly, a closed contour can be deformed freely as long as the deformation does not make $\Gamma$ cross a singularity. To be precise, suppose $\Gamma_A$ and $\Gamma_B$ are closed contours with no singularities between them. Then

\[
\int_{\Gamma_A} f \, dz = \int_{\Gamma_B} f \, dz.
\]

The sketch below illustrates the proof. We insert a small ‘bridge’ of width $\epsilon$ between $\Gamma_A$ and $\Gamma_B$, cutting out a bit of the closed loops as well to get a closed contour

\[
C_\epsilon = \tilde{\Gamma}_A + L^- + (-\tilde{\Gamma}_B) + L^+
\]
as shown below. There are no singularities in this contour (by assumption) so

$$0 = \oint_{C_\epsilon} f \, dz = \int_{\tilde{\Gamma}_A} f \, dz - \int_{\tilde{\Gamma}_B} f \, dz + \int_{L^-} f \, dz + \int_{L^+} f \, dz. \quad (4.4)$$

As $\epsilon \to 0$, the two sides of the bridge approach the same line (but opposite directions) and the cut out parts of the closed loops shrink in size to zero so we have that

$$\tilde{\Gamma}_A \to \Gamma_A, \quad \tilde{\Gamma}_B \to \Gamma_B, \quad L^- \to -L^+. \quad (4.4)$$

Taking these limits in (4.4), the contributions along the bridge cancel, leaving

$$0 = \int_{\Gamma_A} f \, dz - \int_{\Gamma_B} f \, dz.$$

**Key point:** In contour integrals over paths where $f$ is analytic, the contour can be deformed as long as it does not touch/cross a singularity.

In particular, we can deform contours into unions of simple shapes like circular arcs and straight lines, which are nice for computation.

Contour deformation is a powerful tool because if interested in evaluating a contour integral, we can always deform the contour into the simplest form possible:

$$\int_{\Gamma} f \, dz = \int_{\Gamma_{nice}} f \, dz, \quad \Gamma_{nice} = \text{easiest equivalent contour for evaluating the integral.}$$

With a few more tools, ‘bad’ segments of contours can be dealt with, leaving only nice closed contours or contours with simple shapes. The only barriers are the singularities of the function, that block convenient deformation. For this reason, contour integration is more about avoiding singularities than it is about computing integrals.
5. Laurent series; Cauchy’s formula

If a function \( f(z) \) has an isolated pole at \( z_0 \) like \( 1/(z - z_0) \), then a contour around \( f(z) \) can be deformed to arbitrarily small size around the pole. Thus, local behavior of \( f(z) \) is important to know. The basic tool for this is the Laurent series.

**Review:** Recall that the Taylor series for a real function \( f(x) \) around \( x_0 \)

\[
f(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n, \quad a_n = \frac{1}{n!} f^{(n)}(x_0),
\]

which is valid in an interval around \( x_0 \) of radius \( \rho \) (the ‘radius of convergence’)

(5.1) converges for all \( x \) such that \( |x - x_0| < \rho \).

**Complex case:** If \( f(z) \) is analytic in a disk of radius \( \rho \) around \( z_0 \) it has the Taylor series

\[
f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n, \quad a_n = \frac{1}{n!} f^{(n)}(z_0)
\]

(5.2)

which converges in the disk. In particular, this means that the radius of convergence of (5.2) is the distance to the nearest singularity of \( f \).

A Laurent series for \( f \) generalizes Taylor series, allowing an isolated pole. It has the form

\[
f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n
\]

\[
= \frac{a_{-m}}{(z - z_0)^m} + \frac{a_{-m+1}}{(z - z_0)^{m-1}} + \cdots + \frac{a_{-1}}{z - z_0} + a_0 + a_1(z - z_0) + \cdots
\]

where \( m \) is the order of the pole. While the first term \( (n = -m) \) determines the order of the pole, it is the last term \( (n = -1) \) that determines the contour integral (as we will see shortly).

Along with the Taylor series formula, we also have a nice integral form for derivatives:

**Cauchy’s integral formula:** If \( f \) is analytic in a closed contour \( \Gamma \) and \( z_0 \) is inside, then

\[
0 = \oint_{\Gamma} f(z) \, dz \quad \text{(Cauchy’s theorem)}
\]

and for \( n \geq 0 \),

\[
f^{(n)}(z_0) = \frac{n!}{2\pi i} \oint_{\Gamma} \frac{f(z)}{(z - z_0)^{n+1}} \, dz.
\]

**Important case:** Most notably, for \( n = 0 \) the formula reads

\[
f(z_0) = \frac{1}{2\pi i} \oint_{\Gamma} \frac{f(z)}{z - z_0} \, dz,
\]

which is remarkably like the sifting property of the \( \delta \). The formula says that if we know \( f \) on the boundary of a loop, then \( f \) is known **in the entire region**!
6. The residue theorem

The coefficient \(a_{-1}\) is called the **residue** of \(f(z)\) at the pole \(z_0\):

\[
\text{Res}(f, z_0) := a_{-1} = \text{coeff. of } \frac{1}{z - z_0} \text{ in the Laurent series of } f \text{ at } z_0.
\]

For example,

\[
f(z) = \frac{4}{(z - 1)^2} + \frac{3}{z - 1} \implies \text{Res}(f, 1) = 3.
\]

At every other \(z_0\), this function is analytic so there are no other residues. A key result is that this value determines the contour integral of a closed contour around the pole, **regardless of the higher order terms** \((1/(z - z_0)^2 \text{ etc.}):\)

**Residue theorem (one pole):** If \(f\) is analytic in a closed contour \(\Gamma\) except a pole at \(z_0\),

\[
\oint_{\Gamma} f(z) \, dz = 2\pi i \text{Res}(f, z_0).
\]

The integral over the loop only cares about the residue of the pole inside it (nothing else).

**Proof.** The proof illustrates some useful ideas. We can deform the contour to be a circle of arbitrarily small size \(\epsilon \ll 1\) around \(z_0\):

\[
\Gamma_\epsilon = \{ z(t) = z_0 + \epsilon \cdot e^{i\theta}, \quad \theta \in [0, 2\pi] \}.
\]

In this small region, \(f(z)\) is approximately equal to its Laurent series

\[
f(z) \approx \sum_{n=-m}^{n=-1} a_n(z - z_0)^n + (\text{analytic}).
\]

The integral \(\oint \cdots \, dz\) of the analytic part is zero and

\[
\oint_{\Gamma_\epsilon} \frac{1}{(z - z_0)^m} \, dz = \epsilon^{1-m} \int_0^{2\pi} e^{-im\theta} e^{i\theta} \, d\theta = i \int_0^{2\pi} e^{i(1-m)\theta} \, d\theta = \begin{cases} 0 & m \neq 1 \\ 2\pi & m = 1 \end{cases}.
\]

Thus the contour integral only picks up the \(n = -1\) term, and so

\[
\oint_{\Gamma} f(z) \, dz \approx \oint_{\Gamma_\epsilon} f(z) \, dz = 2\pi i a_{-1}.
\]

Finally, take \(\epsilon \to 0\) (shrinking the circle to arbitrarily small) to make the \(\approx\) an equality.
The brute force method to calculate the residue (shortcuts to follow) is to find the Laurent series around \( z = 1 \); this is most easily done by expanding simpler parts and doing ‘power series arithmetic’ (products etc. of series). Two examples:

**Example 1:** Suppose

\[
f(z) = \frac{e^{2z}}{(z-1)^2}
\]

and let \( \Gamma \) be a circle of radius 2 centered at the origin. Then \( z_0 = 1 \) is inside the contour, so

\[
\oint_{\Gamma} f(z) \, dz = 2\pi i \text{Res}(f, 1).
\]

The denominator is okay already, so expand the numerator:

\[
e^{2z} = e^2 e^{2(z-1)} = e^2 (1 + 2(z-1) + 2(z-1)^2 + \cdots).
\]

This gives the Laurent series

\[
f(z) = \frac{e^2}{(z-1)^2} + \frac{2e^2}{z-1} + \cdots
\]

\[
\Rightarrow \oint_{\Gamma} \frac{e^{2z}}{(z-1)^2} \, dz = 2\pi i (2e^2) = 4e^2\pi i.
\]

If instead \( \Gamma \) is a a circle of radius 1/2 around 0, then there is no singularity inside, so

\[
\oint_{|z|=1/2} \frac{e^{2z}}{(z-1)^2} \, dz = 0.
\]

**Example 2:** If there is no residue (no \( 1/(z-z_0) \) term in the Laurent series) the integral is zero, just like for an analytic function. For example,

\[
f(z) = \frac{1}{1 - \cos z} = \frac{1}{\frac{1}{2}z^2 - \frac{1}{4}z^4 + \cdots} = \frac{2}{z^2} \left( \frac{1}{1 - \frac{1}{2}z^2 + \cdots} \right) = \frac{2}{z^2} + 1 + 1 + az^2 + \cdots
\]

which has a pole at \( z = 0 \) but no \( 1/z \) term, so

\[
\oint_{|z|=1} \frac{1}{1 - \cos z} \, dz = 0.
\]

On the other hand,

\[
\oint_{|z|=1} \frac{z}{1 - \cos z} \, dz = \oint_{|z|=1} \frac{2}{z} + \cdots \, dz = 4\pi i,
\]
6.1. **Multiple singularities.** Now suppose $\Gamma$ is a closed contour and $f$ is analytic inside $\Gamma$ except at a set of isolated poles $z_1, z_2, \cdots, z_n$, e.g.

$$f(z) = \frac{1}{(z - 1)(z - 2i)(z + 2)}.$$ 

The contour can be deformed around the singularities into small arcs $C_k$ around each pole and bridges $L_k^\pm$ made of straight lines between successive poles. The idea is sketched below.

Because the full contour is closed, each straight part $L_k^+$ has a ‘return trip’ $L_k^+ \approx -L_k$. Since $f$ is analytic in this region, the contributions cancel due to the rule that

$$\int_{\Gamma} f \, dz = -\int_{-\Gamma} f \, dz.$$ 

Only the circular parts remain, leaving (after taking the width of the bridges to zero)

$$\int_{\Gamma} f \, dz = \sum_{k=1}^{n} \int_{C_k} f \, dz$$

where $C_k$ is a small circular path containing $z_k$. Applying the theorem for one pole, we get:

**Residue theorem:** Let $\Gamma$ be a closed contour and suppose $f$ is analytic inside $\Gamma$ except at an isolated set of poles $z_1, \cdots, z_n$. Then

$$\oint_{\Gamma} f(z) \, dz = 2\pi i \sum_{k=1}^{n} \text{Res}(f, z_k).$$

The residue theorem is powerful because it asserts that if $f$ is mostly analytic (only singular at isolated points), then closed contour integrals only care about the residues at the poles and nothing else. Two examples:
Example 1: Let $\Gamma_R$ is a circle of radius $R$ (counter-clockwise) around the origin. Then

$$\oint_{\Gamma_R} \frac{1}{z-1} + \frac{3}{z+2i} \, dz = \begin{cases} 
0 & R < 1 \\
2\pi i & 1 < R < 2 \\
2\pi i + 6\pi i & 2 < R
\end{cases}$$

since the poles are at $z = 1$ and $z = 3$ with residues 1 and 3, respectively.

Example 2: Let

$$I = \oint_{|z|=1} \frac{e^{2z}}{\sin(3z)} \, dz.$$ 

There are singularities at $z_0 = 0$ and $z = \pm \pi/3$ (plus more). Only $z_0 = 0$ is instead the contour, so we have

$$I = 2\pi i \text{Res}(f, 0).$$

To find the residue, we proceed directly (see later for a shortcut), expanding the numerator/denominator in a Laurent series:

$$\frac{e^{2z}}{\sin(3z)} = \frac{1 + 2z + \cdots}{3z + \cdots} = \frac{1 + 2z + \cdots}{3z (1 + \cdots)} = \frac{1}{3z} + \text{Taylor series}$$

so the residue is $1/3$ and thus

$$I = \oint_{|z|=1} \frac{e^{2z}}{\sin(3z)} \, dz = \frac{2\pi i}{3}.$$ 

On the other hand,

$$I = \oint_{|z|=2} \frac{e^{2z}}{\sin(3z)} \, dz = 2\pi i \left( \text{Res}(f, -\pi/3) + \text{Res}(f, 0) + \text{Res}(f, \pi/3) \right)$$

which would require more work (we really do want the shortcut here; see next page).
6.2. **Simple poles (shortcut):** Let \( f(z) = p(z)/q(z) \) (a common form). Suppose \( p(z) \) and \( q(z) \) have no singularities inside a closed contour \( \Gamma \) and \( q(z) = (z - z_0)r(z) \) (a common form). Suppose \( p(z) \) and \( q(z) \) have no singularities inside a closed contour \( \Gamma \) and \( q(z) = (z - z_0)r(z) \), \( r(z) \neq 0 \) since otherwise the root would be a double root or higher. It follows that

\[
I = \oint_{\Gamma} f(z) \, dz = \oint_{\Gamma} \frac{p(z)}{z - z_0} \, dz = 2\pi i \frac{p(z_0)}{r(z_0)}
\]

since \( p(z)/r(z) \) has a Taylor series in \( z - z_0 \) (no singularities) and the first term is \( p(z_0)/r(z_0) \).

Obtaining \( r(z) \) might be a nuisance, but we can shortcut this by observing that

\[
q'(z) = (z - z_0)r(z) + r(z) \implies q'(z_0) = r(z_0).
\]

**Shortcut (simple pole):** If \( q \) has a simple zero at \( z_0 \) (and no others) in \( \Gamma \) and \( p, q \) have no singularities in \( \Gamma \) then

\[
\oint_{\Gamma} \frac{p(z)}{q(z)} \, dz = 2\pi i \frac{p(z_0)}{q'(z_0)}
\]

i.e. the residue of \( f = p/q \) at \( z_0 \) is \( \text{Res}(f, z_0) = p(z_0)/q'(z_0) \).

Returning to the earlier example of computing

\[
\oint_{|z|=2} \frac{e^{2z}}{\sin 3z} \, dz
\]

we have

\[
f(z) = \frac{e^{2z}}{\sin 3z} \implies p = e^{2z}, \quad q = \sin 3z \implies \text{Res}(f, 0) = p(0)/q'(0) = 1/3.
\]

Similarly, at the two other poles in the contour,

\[
\text{Res}(f, \pm \pi/3) = p(\pm \pi/3)/q'(\pm \pi/3) = \frac{e^{\pm 2\pi/3}}{3\cos(\pm \pi)} = -\frac{1}{3} e^{\pm 2\pi/3}.
\]

It follows that for the circle of radius 2,

\[
\oint_{|z|=2} \frac{e^{2z}}{\sin 3z} \, dz = 2\pi i \left( \frac{1}{3} e^{-2\pi/3} + \frac{1}{3} e^{2\pi/3} \right) = \frac{2\pi i}{3} (e^{-2\pi/3} + 1 + e^{2\pi/3}).
\]

6.3. **Another approach:** We can also use Cauchy’s integral formula,

\[
2\pi i f(z_0) = \oint_{\Gamma} \frac{f(z)}{z - z_0} \, dz
\]

to calculate residues (this formula holds if \( f \) is analytic inside \( \Gamma \)). For instance,

\[
\oint_{|z|=2} \frac{\sin 2z}{z - 1} \, dz = 2\pi i \sin(2).
\]
This is most useful if the integrand is of the form \( \frac{\text{analytic}}{z - z_0} \). More generally,

\[
\frac{2\pi if^{(n)}(z_0)}{n!} = \oint_\Gamma \frac{f(z)}{(z - z_0)^{n+1}} dz
\]

which is useful for higher-order poles in this form, e.g.

\[
\oint_{|z|=2} \frac{\sin 2z}{(z - 1)^3} dz = -\frac{2\pi i}{6} \cdot 2^3 \sin 2 = -\frac{8\pi i}{3} \sin 2.
\]

If the \((z - z_0)^k\) factor is not already there, then we must do some work to use Cauchy’s integral formula since the \(f(z)\) (numerator) must be analytic (see below).

**General rule:** Suppose \( f(z) = p(z)/q(z) \), that \( p(z) \) is analytic and \( q \) has a zero of order \( m \) at \( z_0 \). Then

\[
q(z) = (z - z_0)^m r(z), \quad r(z_0) \neq 0.
\]

Letting \( s(z) = p(z)/r(z) \) (which is analytic near \( z_0 \), we have

\[
f(z) = \frac{s(z)}{(z - z_0)^m}.
\]

Now let \( \Gamma \) be a closed contour around \( z_0 \), enclosing only the singularity at \( z_0 \). From the residue theorem and Cauchy’s integral formula,

\[
2\pi i \text{Res}(f, z_0) = \oint_\Gamma f(z) \, dz = \oint_\Gamma \frac{s(z)}{(z - z_0)^m} \, dz = 2\pi i \frac{s^{(m-1)}(z_0)}{(m-1)!}.
\]

This gives the contour integral around \( z_0 \) and also a formula for the residue:

\[
\text{Res}(f, z_0) = \frac{1}{(m-1)!} \lim_{z \to z_0} \left[ \frac{d^{m-1}}{dz^{m-1}} \left( (z - z_0)^m f(z) \right) \right]
\]

after plugging in \( f = \frac{p}{q} = \frac{s}{(z - z_0)^m} \). The limit is there because one may need to apply L’Hôpital’s rule to evaluate the expression (can be 0/0 at \( z_0 \)).