Reading: Read and study the following:

- [suggested:] Chapter 9.2-9.3; 9.4.2 (not 9.4.3)
- Calculus review document (to be used for multi-d PDEs and also here for P2). We will make frequent use of some of the theorems and the spherical/cylindrical coordinates, and occasional use of the identities.

Problems:

- Chapter 9: 9.3.14c and 9.4.3 (Suggested [don’t turn in]: also 9.3.12a)
- Problems P1 to P5 below.
- Review spherical/cylindrical coordinates (no problems to do here, but you could re-derive some of the formulas on the review sheet, e.g. gradient in polar coordinates; all are useful exercises but some are more tedious than others).

PROBLEMS

P1 (some calculus for later use). a) Let \( u(x) \) and \( v(x) \) be (scalar) functions on \( \mathbb{R}^n \) and let \( \Omega \subset \mathbb{R}^n \) be a bounded region. Use integration by parts to derive Green’s formula

\[
\int_{\Omega} u \nabla^2 v - v \nabla^2 u \, dV = \int_{\partial \Omega} u \frac{\partial v}{\partial n} - v \frac{\partial u}{\partial n} \, dS
\]

where \( n \) is the outward normal to \( \Omega \) and \( \partial \Omega \) is the boundary. What is the result for \( n = 1 \)?

b) Consider the Neumann problem for Laplace’s equation in a rectangle \( \Omega = [0,1] \times [0,2] \),

\[
\nabla^2 u = f(x,y) \text{ in } \Omega
\]

\[
 u_y(x,0) = 0, \quad u_y(x,2) = g(x), \quad u_x(0,y) = h(y), \quad u_x(1,y) = 0.
\]

Integrate the PDE over the rectangle \( \Omega = [0,1] \times [0,2] \) and use (a) with \( v = 1 \) to derive a condition on \( f, g, h \) required for a solution to exist.

c) Let \( \hat{\rho} \) be the radial unit vector in spherical coordinates \((\rho, \theta, \phi)\) and consider the field

\[
v(\rho, \theta, \phi) = \frac{1}{\rho^2} \hat{\rho}.
\]

Let \( \Omega \) be a sphere of radius 1 and let \( \Omega_\epsilon \) be this sphere with an inner core of radius \( \epsilon \) removed:

\[
\Omega_\epsilon = \{(\rho, \theta, \phi) : \epsilon \leq \rho \leq 1\}.
\]

Show (by direct computation) that divergence theorem holds for \( \epsilon > 0 \). What about \( \epsilon = 0 \)?
P2. A metal beam is attached at $x = 0$ to a wall and free at $x = 1$, pulled down by a gravitational force $f(x)$. Its deflection $u(x)$ from horizontal satisfies

$$ku^{(4)}(x) = f(x), \quad u(0) = u'(0) = 0, \quad u''(1) = u'''(1) = 0$$

where $k > 0$ is a parameter.\(^1\) Let $k = 1$ for convenience.

a) Determine the form of the (piecewise) Green’s function $g(x, s)$ and write down the linear system to solve for the coefficients. \textit{Hint: by a good choice of basis you can simplify a bit.}

b) Solve the system in (a) to find the Green’s function. \textit{Hint: this may be messy, depending on (a); use computer algebra if necessary.}

---

P3 (separable FIE). Define the operator (on $L^2[0, 1]$)

$$Lu = \int_0^1 (x - t)^2 u(t) dt.$$ 

b) Derive the linear system to solve for the coefficients $c_j$ of the eigenfunctions $\phi = \sum c_j \alpha_j$ and compute the non-zero eigenvalues (numerically is fine).

c) Find a condition on $f$ that guarantees a solution exists for $Lu = f$.

---

P4. Consider a steady state problem for diffusion in a ring,

$$-u_{\theta\theta} - 4u = f(\theta), \quad u(\theta) \text{ } 2\pi \text{-periodic}, \quad \theta \in [0, 2\pi]$$

where $f(\theta)$ is a source. Use the Fredholm alternative to find the condition on $f$ such that this problem has no solution vs. infinitely many. \textit{Hint: be careful; the problem is not regular.} Do not solve for $u$ here!

---

P5 (Properties of the Dirac delta). Let $\delta(x)$ denote the Dirac delta in $\mathbb{R}$.

a) Make sense of the expression $x\delta(x)$ by computing $\int_a^b x\delta(x) f(x) dx$ with $\delta = H'(x)$.

b) Let $\langle f, g \rangle = \int_{-\infty}^{\infty} f(x) g(x) dx$. Show (using IBP and possibly a bit of hand-waving\(^2\)) that

$$\langle \delta'(x), f(x) \rangle = -\langle \delta(x), f'(x) \rangle = -f'(0)$$

for all functions $f$ such that $f \to 0$ as $x \to \pm \infty$.

c) Let $\delta(x)$ be the Dirac delta in $\mathbb{R}^n$ (see notes). Show that if $a \neq 0$ then the sifting property (and/or unit mass property) implies that

$$\delta(ax) = \frac{1}{|a|^n}\delta(x).$$

\textit{Hint: integrate over $\mathbb{R}^n$; assume the ‘change of variables’ rules apply for $\delta(x)$ in the integral.} In particular, note that $\delta(-x) = \delta(x)$.

---

\(^1\)This is the rigidity of the beam, $k = EI$ where $E$ is Young’s modulus and $I$ is the second moment.

\(^2\)This IBP approach is the way a derivative of a distribution is defined in general, although there are some technical conditions on $f$ to deal with to be rigorous.