1. **Sturm-Liouville theory in 1D**

\[ L^2 \text{ inner product}: \langle f, g \rangle = \int_a^b f(x)g(x) \, dx, \quad L^2[a,b] = \{ f : \langle f, f \rangle < \infty \} \]

Weighted inner product: \[ \langle f, g \rangle_\sigma = \int_a^b f(x)g(x)\sigma(x) \, dx, \quad L^2_\sigma[a,b] = \{ f : \langle f, f \rangle_\sigma < \infty \} \]

A **Sturm-Liouville (SL) operator** in \([a,b]\) (with \( p(x) \geq 0 \)) has the form

\[ Lu = -(p(x)u_x)_x + q(x)u. \]

A **Sturm-Liouville problem (SLP)** for \( L \) and a ‘weight function’ \( \sigma(x) \) has the form

\[ Lu = \lambda \sigma(x)u, \quad x \in (a,b) \text{ plus hom. BCs}. \]

**Green’s formula for** (1):

\[ \langle Lu, v \rangle = p(uv_x - vu_x)_a^b + \langle u, Lv \rangle \]

for all smooth \( u, v \) in \([a,b]\). Moreover, \( L \) regular \( \implies \) self adjoint in the \( L^2 \) inner product:

\[ \langle Lu, v \rangle = \langle u, Lv \rangle \text{ for all } u, v \text{ satisfying the BCs}. \]

**Main theorem:** The SLP (2) for a **self-adjoint** \( L \) has the following properties:

i) The eigenvalues and eigenfunctions are real; the eigenvalues are an infinite sequence with a smallest eigenvalue, tending to \( \infty \):

\[ \lambda_1 < \lambda_2 < \lambda_3 < \cdots \to \infty. \]

ii) The eigenfunctions form a basis for \( L^2_\sigma[a,b] \), and they are orthogonal in the weighted inner product. That is, the eigenfunctions satisfy

\[ \langle \phi_m, \phi_n \rangle_\sigma = \int_a^b \phi_m(x)\phi_n(x)\sigma(x) \, dx = 0, \quad m \neq n \]

Every \( f \in L^2_\sigma[a,b] \) has a unique representation in the basis:

\[ f = \sum_{n=1}^{\infty} a_n \phi_n(x), \quad a_n = \frac{\langle f, \phi_n \rangle_\sigma}{\langle \phi_n, \phi_n \rangle_\sigma}. \]

**Eigenvalue problems** (standard results, 1d):

\[ -\phi'' = \lambda \phi, \quad \phi(0) = \phi(L) = 0 \implies \phi_n = \sin(n\pi x / L), \quad \lambda_n = \left( \frac{n\pi}{L} \right)^2, \quad n \geq 1 \]

\[ -\phi'' = \lambda \phi, \quad \phi'(0) = \phi'(L) = 0 \implies \phi_n = \cos(n\pi x / L), \quad \lambda_n = \left( \frac{n\pi}{L} \right)^2, \quad n \geq 0 \]

\[ -\phi'' = \lambda \phi, \quad \phi(0) = \phi(L) = 0 \implies \phi_n = \sin((n - \frac{1}{2})\pi x / L), \quad \lambda_n = \left( \frac{(n - \frac{1}{2})\pi}{L} \right)^2, \quad n \geq 1 \]

\[ -\phi'' = \lambda \phi, \quad \phi'(0) = \phi(L) = 0 \implies \phi_n = \cos((n - \frac{1}{2})\pi x / L), \quad \lambda_n = \left( \frac{(n - \frac{1}{2})\pi}{L} \right)^2, \quad n \geq 1 \]

\[ -\phi'' = \lambda \phi, \quad \phi 2\pi\text{-periodic} \implies \phi_n = \cos nx \text{ or } \sin nx, \quad \lambda_n = n^2, \quad n \geq 0 \]
2. EIGENVALUE/COEFFICIENT ODEs (separable problems)

**Bessel equation:** for $R(r)$ (disk, cylinder), $\lambda > 0$ (for $\lambda < 0$ use Modified Bessel)

$$R'' + \frac{1}{r} R' + (\lambda - \frac{\nu^2}{r^2}) R = 0, \quad \lambda > 0 \implies y = c_1 J_\nu(x \sqrt{\lambda}) + c_2 Y_\nu(x \sqrt{\lambda})$$

- For $\lambda = 0$, use Cauchy-Euler procedure instead (bottom of page)
- Values at zero: $J_0(0) = 1$ and $J_\nu(0) = 0$ for $\nu > 0$, $|Y_\nu(0)| = \infty$ (unbounded)
- Zeros: $J_\nu(z)$ has positive zeros $\gamma_{\nu,n}$: $0 < \gamma_{\nu,1} < \gamma_{\nu,2} < \cdots \to \infty$
- Zeros: $J'_\nu(z)$ has positive zeros $\gamma'_{\nu,n}$: $0 < \gamma'_{\nu,1} < \gamma'_{\nu,2} < \cdots \to \infty$

**Spherical Bessel equation:** For $R(r)$ (sphere),

$$\frac{1}{r^2}(r^2 R')' + (\lambda - \frac{\nu(\nu + 1)}{x^2}) R = 0 \implies R = c_1 \frac{J_{\nu+1/2}(r \sqrt{\lambda})}{\sqrt{r}} + c_2 \frac{Y_{\nu+1/2}(r \sqrt{\lambda})}{\sqrt{r}}$$

$J$ term bounded in $[0, a]$ and $Y$ term unbounded in $[0, a]$ ($\infty$ at $r = 0$)

**Spherical harmonics:** Eigenfunctions $Y^n_m(\theta, \phi)$ for $-\nabla^2 Y = \lambda Y$ (sphere surface), where

$$\nabla^2 Y = \frac{1}{\sin \phi} (\sin \phi Y_\phi)' + \frac{1}{\sin^2 \phi} Y_{\theta \theta}$$

- Related to full Laplacian $\nabla^2 u(r, \theta, \phi)$ by $\nabla^2 u = \frac{1}{r^2} (r^2 u_r)_r + \frac{1}{r^2} \nabla^2 u$.
- Separated, $Y = g(\theta)h(\phi)$ with $-g'' = m^2 g$, $m \geq 0$.

**\(\phi\)-dir:**

$$\frac{1}{\sin \phi} (\sin \phi h')' + \left(\lambda - \frac{m^2}{\sin^2 \phi}\right) h = 0$$

**Transformed:**

$$((1 - \xi^2) y')' + (\lambda - \frac{m^2}{1 - \xi^2}) y = 0, \quad \xi = \cos \phi, y(\xi) = h(\phi)$$

**Solutions:**

$$\left\{ \begin{align*} Y^n_m &= P^m_n(\cos \phi)(\cos m\theta \text{ or } \sin m\theta), \\
\lambda_n &= n(n+1) \text{ (independent of } m) \end{align*} \right. \quad 0 \leq m \leq n$$

**Modified Bessel functions:** (Bessel, opposite sign), $\eta > 0$

$$y'' + \frac{1}{x} y' - (\eta + \frac{\nu^2}{x^2}) y = 0, \quad \implies y = c_1 I_\nu(x \sqrt{\eta}) + c_2 K_\nu(x \sqrt{\eta})$$

- Zeros: None; both $K_\nu(z)$ and $I_\nu(z)$ are positive for $z > 0$
- Values at zero: $I_0(0) = 1$ and $I_\nu(0) = 0$ for $\nu > 0$, $K_\nu(0) = \infty$ (unbounded)

**Cauchy-Euler equations:** $p, q$ are constants. Three cases, depending on roots of $p(r)$.

$$x^2 y'' + xpy' + qy = 0, \quad x > 0 \implies x^\alpha \text{ is a solution} \iff 0 = p(\alpha) = \alpha(\alpha - 1) + p\alpha + q.$$

- $\alpha_1 \neq \alpha_2$, real $\implies y = c_1 x^{\alpha_1} + c_2 x^{\alpha_2}$
- $\alpha_1 = \alpha_2 \implies y = c_1 x^{\alpha_1} + c_2 x^{\alpha_1} \log x$
- $\alpha = s \pm \omega i \implies y = c_1 x^s \cos(\omega \log x) + c_2 x^s \sin(\omega \log x)$. 
3. Calculus/Complex Variables

\[ \int \ldots dV = \text{volume integral (over domain)} \] and \[ \int \ldots dS = \text{integral over boundary} \]

\( \Omega = \text{domain and } \partial \Omega = \text{boundary, } \mathbf{n} = \text{outward normal, } \partial u / \partial \mathbf{n} = \nabla u \cdot \mathbf{n} = \text{normal derivative} \)

**Divergence thm:** \[ \int_{\Omega} \nabla \cdot \mathbf{v} \, dV = \int_{\partial \Omega} \mathbf{v} \cdot dS. \]

**Int. by parts:** \[ \int_{\Omega} f \nabla^2 g \, dV = \int_{\partial \Omega} f \frac{\partial g}{\partial n} \, dS - \int_{\Omega} \nabla f \cdot \nabla g \, dV. \]

**Green’s formula:** \[ \int_{\Omega} (f \nabla^2 g - g \nabla^2 f) \, dV = \int_{\partial \Omega} f \frac{\partial g}{\partial n} - g \frac{\partial f}{\partial n} \, dS. \]

**Cylindrical:** radius \( r \), angle \( \theta \) (in \( xy \) plane) and \( z \)

**Unit vectors:** \( \hat{r} = \cos \theta \hat{x} + \sin \theta \hat{y}, \quad \hat{\theta} = -\sin \theta \hat{x} + \cos \theta \hat{y}, \quad \hat{z} = \hat{z} \)

**volume:** \( dV = r \, dr \, d\theta \, dz \),** surface (rad. a):** \( dS = a \, dz \, d\theta \)

**Laplacian:** \( \nabla^2 f = \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial f}{\partial r} \right) + \frac{1}{r^2 \sin^2 \phi} \frac{\partial^2 f}{\partial \theta^2} + \frac{1}{r^2 \sin \phi} \frac{\partial}{\partial \phi} \left( \sin \phi \frac{\partial f}{\partial \phi} \right) \)

**Spherical:** radius \( r \), azimuthal angle \( \theta \), polar angle \( \phi \)

**Coordinates:** \[
\begin{align*}
    x &= r \sin \phi \cos \theta, \\
    y &= r \sin \phi \sin \theta, \\
    z &= r \cos \phi
\end{align*}
\]

**volume:** \( dV = r^2 \sin \phi \, dr \, d\theta \, d\phi \),** surface (rad. a):** \( dS = a^2 \sin \phi \, d\phi \, d\theta \)

**Laplacian:** \( \nabla^2 f = \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial f}{\partial r} \right) + \frac{1}{r^2 \sin^2 \phi} \frac{\partial^2 f}{\partial \theta^2} + \frac{1}{r^2 \sin \phi} \frac{\partial}{\partial \phi} \left( \sin \phi \frac{\partial f}{\partial \phi} \right) \)

4. Complex Variables

Cauchy-Riemann eqs.: \( u_x = v_y, \quad u_y = -v_x \) for \( f(z) = u + iv \)

Cauchy integral formula: \( f^{(n)}(z_0) = \frac{n!}{2\pi i} \int_{\Gamma} \frac{f(z)}{(z-z_0)^{n+1}} \, dz, \quad n \geq 0 \)

Principal value: \( \text{PV} \int_{-\infty}^{\infty} \frac{f(x)}{x-x_0} \, dx = \lim_{\epsilon \downarrow 0} \left( \int_{-\infty}^{-\epsilon} \frac{f(x)}{x-x_0} \, dx + \int_{\epsilon}^{\infty} \frac{f(x)}{x-x_0} \, dx \right) \).

**Common Taylor/Laurent series**

\[
\begin{align*}
    e^z &= 1 + z + \frac{1}{2} z^2 + \frac{1}{3!} z^3 + \cdots \\
    \frac{1}{1-z} &= 1 + z + z^2 + \cdots \\
    \cot z &= \frac{1}{z} - \frac{z}{3} - \frac{1}{45} z^3 - \cdots \\
    \sin z &= z - \frac{1}{3!} z^3 + \frac{1}{5!} z^5 - \cdots \\
    \cos z &= 1 + \frac{1}{2} z^2 + \frac{1}{4!} z^4 + \cdots \\
    \tan z &= z + \frac{z^3}{3} + \frac{2}{15} z^5 + \cdots 
\end{align*}
\]
5. Transforms

Fourier transform \((f(x) \rightarrow F(k))\)

common functions:
\[ e^{-ax^2} \rightarrow \frac{1}{\sqrt{4\pi a}} e^{-k^2/(4a)}, \]
\[ \sqrt{\pi/b} e^{-x^2/(4b)} \rightarrow e^{-bk^2} \]
\[ \delta(x - x_0) \rightarrow \frac{1}{2\pi} e^{ikx_0} \]
\[ \frac{2a}{x^2 + a^2} \rightarrow e^{-a|k|} \]

transform rules:
\[ \mathcal{F}^{-1}(F) = \int_{-\infty}^{\infty} F(k)e^{-ikx} dk \]
\[ \frac{1}{2\pi} \int_{-\infty}^{\infty} f(y)g(x - y) dy \rightarrow F(k)G(k) \]
\[ f(x - a) \rightarrow e^{iak}F(k) \]
\[ e^{-iax}f(x) \rightarrow F(k - a) \]
\[ \partial f/\partial x \rightarrow -ikF(k) \]

Sine and cosine transforms:

Sine transforms:
\[ S[f] = \frac{2}{\pi} \int_{0}^{\infty} f(x) \sin kx \, dx, \]
\[ S^{-1}[F] = \int_{0}^{\infty} F(k) \sin kx \, dk \]
\[ S[e^{-ax}] \rightarrow \frac{2}{\pi} \frac{k}{a^2 + k^2} \]
\[ S[\frac{x}{x^2 + a^2}] \rightarrow e^{-ak} \]
\[ S[1] \rightarrow \frac{2}{\pi k} \]
\[ S[f'(x)] \rightarrow -kC[f(x)] \]
\[ S[f''(x)] \rightarrow \frac{2}{\pi} k f(0) - k^2 F(k) \]

Cosine transforms:
\[ C[f] = \frac{2}{\pi} \int_{0}^{\infty} f(x) \cos kx \, dx, \]
\[ C^{-1}[F] = \int_{0}^{\infty} F(k) \cos kx \, dk \]
\[ C[e^{-ax}] \rightarrow \frac{2}{\pi} \frac{a}{a^2 + k^2} \]
\[ C[\frac{a}{x^2 + a^2}] \rightarrow e^{-ak} \]
\[ C[1] \rightarrow \frac{1}{\pi a} e^{-k^2/(4a)} \]
\[ C[f'(x)] \rightarrow -\frac{2}{\pi} f(0) + kS[f(x)] \]
\[ C[f''(x)] \rightarrow -\frac{2}{\pi} f'(0) - k^2 F(k) \]

Laplace transform: \((f(t) \rightarrow F(s))\)

\[ \frac{d^n f}{dt^n} = -f^{(n-1)}(0) - s f^{(n-2)}(0) - \cdots - s^{n-1} f(0) + s^n F(s) \]

common functions:
\[ t^n \quad (n > -1) \rightarrow n!s^{-(n+1)} \]
\[ e^{at} \rightarrow 1/(s - a) \]
\[ \sin at \rightarrow \frac{a}{s^2 + a^2} \]
\[ \delta(t - a) \rightarrow e^{-as} \]
\[ H(t - a) \rightarrow e^{-as}/s \]
\[ e^{-a\sqrt{t}} \rightarrow \frac{a}{\sqrt{4\pi t^{3/2}}} e^{-a^2/4t} \]

transform rules:
\[ \mathcal{L}^{-1}(F) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} F(s)e^{st} \, ds \]
\[ df/dt \rightarrow -f(0) + sF(s) \]
\[ -tf(t) \rightarrow dF/ds \]
\[ \int_{0}^{t} f(t - t_0)g(t_0) \, dt_0 \rightarrow F(s)G(s) \]
\[ H(t - a)f(t - a) \rightarrow e^{-as}F(s) \quad (a > 0) \]