SOME ADDITIONAL REVIEW PROBLEMS
MATH 551, FALL 2019

(NOT A HOMEWORK; DO NOT TURN IN!

Notes: The following problems are intended for review. They supplement the existing material, focusing on complex analysis. For the rest, the starred book problems are suggested (and review notes, old homework).

Not all problems here are ‘exam-like’; some are more complete solutions, a subset of which might appear on an exam.

1. Complex variables

R1. Let
\[ f(z) = \frac{1}{z^5 + 13z + 4}. \]
What is the sum of the residues of \( f(z) \)? *Hint: consider a circle of radius \( R \).*

R2. Evaluate (with a contour integral)
\[ I = \int_0^\infty f(x) \, dx \text{ where } f(x) = \frac{x}{x^4 + 1}. \]
*Hint: use a sector of a circle.*

R3. Let \( f(z) \) be an analytic function and let
\[ g(z) = \frac{1}{z^2} \cot \pi z. \]
Find the sum of the residues of \( g(z) \) (leave your answer in the form of a sum). You may use the fact that
\[ \cot z = \frac{1}{z} - \frac{1}{3} z - \frac{45}{z^3} + \cdots. \]
Note: you don’t need to do the argument in R1 here (it’s harder since a circle does not work).

R4. Use a contour integral to evaluate
\[ f(t) = \frac{1}{2\pi i} \int_{1-i\infty}^{1+i\infty} \frac{e^{-st}}{s^m} \, ds, \quad t > 0. \]
where \( m \) is a positive integer. You may use the fact that
\[ e^z = \sum_{n \geq 0} \frac{1}{n!} z^n. \] Don’t use Laplace transform properties here.
R5. Use an indented contour ($\epsilon$-sized semi-circle around 0) to compute
\[ \mathcal{F}^{-1}\left(\frac{1}{k}\right) = \int_{-\infty}^{\infty} \frac{1}{k} e^{-ikx} \, dk. \]
Assume that the ‘$R \to \infty$’ limits on semi-circles vanish (but not $\epsilon \to 0$).

R6. Compute the inverse Laplace transform of \[ F(s) = \frac{1}{(s^2 + 1)^2} \]
(a) using direct inversion (with a contour integral) and (b) using the convolution theorem (or other properties). Also find an ODE IVP for which $\mathcal{L}^{-1}(F)$ is the solution.

R7. Use a box contour (Hint: the box is ‘vertical’ here) to evaluate
\[ \int_{-i\infty}^{i\infty} \frac{1}{\cos 2z} \, dz. \]

R8 (Cosine transform). Use even extension to $(-\infty, \infty)$ to compute the solution $u = f \ast h$ to the following problem:
\[
\begin{align*}
    u_t &= ku_{xx}, \quad x \in (0, \infty), \\
    u &\to 0 \text{ as } x \to \infty \\
    u_x(0, t) &= 0 \\
    u(x, 0) &= f(x)
\end{align*}
\]
Your answer should be in the form of a convolution with a function $h$. It may help to have \[ G(x) = \frac{1}{\sqrt{4\pi t}} e^{-x^2/4t}. \]

R9 (Laplace transform). Use the Laplace transform to find the solution to
\[
\begin{align*}
    u_t &= 4u_{xx}, \quad x \in (0, \infty), \\
    u &\to 0 \text{ as } x \to \infty \\
    u(0, t) &= g(t) \\
    u(x, 0) &= 0, \quad u_t(x, 0) = 0
\end{align*}
\]

R10 (variant of 13.5.6). Consider the following problem in a bounded interval,
\[
\begin{align*}
    u_t &= u_{xx}, \quad x \in (0, 2), \\
    u(0, t) &= 0, \quad u_x(2, t) = f(t) \\
    u(x, 0) &= g(x)
\end{align*}
\]
a) Use the Laplace transform to find the solution.

b) Compare this to the solution using eigenfunction expansion.
2. Partial solutions

Not all solutions are complete; some steps omitted.

R1 (solution). Let $C_R$ be a (counter-clockwise) circle of radius $R$. Then
\[
\left| \oint_{C_R} f(z) \, dz \right| \leq 2\pi R \frac{1}{R^5 - 13R - 4} \sim \frac{2\pi}{R^4}
\]
which goes to zero as $R \to \infty$. When $R$ is large enough, $C_R$ contains all residues of $f$, so
\[
2\pi i \sum_k \text{Res}(f; z_k) = \oint_{C_R} f(z) \, dz \to 0 \text{ as } R \to \infty.
\]
Thus the sum of the residues must be zero.

R2 (solution). Let $\Gamma_R$ be the ray from 0 to $R$ on the real axis, let $\Gamma_R^-$ be the return path from $iR$ to 0 on the imaginary axis and let $C_R$ be the sector of radius $R$ from $\theta = 0$ to $\theta = \pi/2$. Then (parametrize: $z(t) = (R-t)i$)
\[
\int_{\Gamma_R^-} f(z) \, dz = \int_0^R \frac{it}{(it)^4 + 1}(-i \, dt) = \int_0^R \frac{t}{t^4 + 1} \, dt = I.
\]
Now estimate the integral on $C_R$:
\[
\left| \int_{C_R} \frac{z}{z^4 + 1} \, dz \right| \leq \frac{\pi R}{2} \max_{z \in C_R} \left| \frac{z}{z^4 + 1} \right| \leq \frac{\pi R}{2(R^4 - 1)} \sim \frac{\pi}{2R^3} \to 0 \text{ as } R \to \infty.
\]
It follows, integrating over the closed contour, that
\[
2\pi i \sum_k \text{Res} = \oint f(z) \, dz = \left( \int_{\Gamma_R} + \int_{\Gamma_R^-} + \int_{C_R} \right) f(z) \, dz \to I + I + 0 \text{ as } R \to \infty.
\]
The residues are where $z^4 = -1$ so at $z = e^{i\theta}$, $\theta = (2k + 1)\pi/4$ for $k = 0, 1, 2, 3$. Of these, only $e^{i\pi/4}$ is inside the sector, so
\[
I = \frac{2\pi i}{2} \frac{e^{i\pi/4}}{4e^{3i\pi/4}} = \pi i \frac{1}{4(-i)} = \frac{\pi}{4}.
\]

R3 (solution). The residues are at $z_0 = 0$ (not simple) and simple poles at $z_k = k$ for integers $k \neq 0$. For the simple poles,
\[
\text{Res}(g; z_k) = \text{Res}\left( \frac{\cos \pi z / z^2}{\sin \pi z}; z_k \right) = \frac{1}{\pi z_k^2} = \frac{1}{\pi k^2}
\]
using $p = \cos \pi z / z^2$ and $q = \sin \pi z$. For the pole at $z = 0$, use the given Laurent series:
\[
g(z) = \frac{1}{\pi z^3} - \frac{\pi}{3z} + O(z) \implies \text{Res}(g; 0) = -1/3.
\]
Thus the sum of the residues is
\[
-\frac{\pi}{3} + 2 \sum_{k=1}^\infty \frac{1}{\pi k^2}.
\]
Remark (extra): With effort, one can use a box contour around these poles to derive

\[ 0 = \oint_{\Gamma} g(z) \, dz = 2\pi i \sum \text{Res} \Rightarrow \sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6}, \]

one of the many proofs of this fact about the Riemann zeta function \( \zeta(2) = \pi^2/6 \).

R4 (solution). If \( m = 1 \) the Jordan Curve lemma is required. Otherwise, the ML estimate works. Close the contour in the usual way (as in the examples in the Lecture notes) for \( L^{-1} \).

The only residue is at \( s = 0 \). The Laurent series is

\[ \frac{e^{-st}}{s^m} = \frac{1}{s^m} \left( \cdots + \frac{1}{(m-1)!} t^{m-1} s^{m-1} + \cdots \right) = \cdots + \frac{1}{(m-1)!} \frac{t^{m-1}}{s} + \cdots \]

where the \( \cdots \) indicates higher-order poles and Taylor series terms. This gives

\[ \frac{1}{(m-1)!} t^{m-1} = \frac{1}{2\pi i} \oint_{C_R} \frac{e^{-st}}{s^m} \, ds = \int_{C_R} \cdots ds + f(t) \to f(t) \text{ as } R \to \infty. \]

R5 (solution). This problem is almost the same as the indented contour example in the lecture notes. The (small) difference is that the answer depends on \( x \) due to the choice of semi-circle. The result is

\[ \int_{-\infty}^{\infty} \frac{1}{k} e^{-ikx} \, dk = -i\pi \begin{cases} -1 & k < 0 \\ 1 & k > 0 \end{cases} \]

usually written \( \mathcal{F}^{-1}(1/k) = -i\pi \operatorname{sgn}(k) \).

R6 (solution). To compute directly, consider

\[ I = \frac{1}{2\pi i} \int_{C-e^{i\infty}}^{C+e^{i\infty}} \frac{e^{st}}{(s^2+1)^2} \, ds. \]

Let \( q(s) = (s^2 + 1)^2 \). By the argument in the lecture notes, the left semi-circle \( C_R \) is the correct one to use; then

\[ \left| \int_{C_R} \frac{e^{st}}{(s^2+1)^2} \, ds \right| \leq \pi R \frac{1}{(R^2-1)^2} \sim \frac{\pi}{R^3} \text{ as } R \to \infty. \]

All the residues are inside the contour, so it follows that

\[ I = \sum \text{Res}(F; z_k) = \text{Res}(\frac{e^{st}}{q}; i) + \text{Res}(\frac{e^{st}}{q}; -i). \]

Both poles are order 2. It’s easiest to use Cauchy’s integral formula,

\[ \oint_{\Gamma} \frac{f(z)}{(z-z_0)^{n+1}} \, dz = \frac{2\pi i}{n!} f^{(n)}(z_0). \]

Note that since \( \Gamma \) here is a small contour around \( z_0 \), the above also equals \( 2\pi i \) times the residue of the integrand at \( z_0 \). It follows that if \( f(z) \) is analytic then

\[ \text{Res}(f/(z-z_0)^2; z_0) = f'(z_0). \]
Using this rule with \( f = \frac{e^{st}}{(s + i)^2} \) at \( s = i \) and \( f = \frac{e^{st}}{(s - i)^2} \) at \( s = -i \), we get
\[
\text{Res}(\frac{e^{st}}{q}; i) = \frac{d}{ds} \left( \frac{e^{st}}{(s + i)^2} \right) \bigg|_{s=i} = -\frac{2e^{it}}{(2i)^3} + \frac{te^{it}}{(2i)^2},
\]
\[
\text{Res}(\frac{e^{st}}{q}; -i) = \frac{d}{ds} \left( \frac{e^{st}}{(s - i)^2} \right) \bigg|_{s=-i} = -\frac{2e^{-it}}{(-2i)^3} + \frac{te^{-it}}{(-2i)^2}.
\]
Adding them together and combining the ± exponentials,
\[
I = \frac{1}{2} \sin t - \frac{1}{2} t \cos t.
\]

**Via convolution:** Let \( G(s) = 1/(s^2 + 1) \), for which \( \mathcal{L}^{-1}(G) = \sin t \). Then
\[
\mathcal{L}^{-1}\left( \frac{1}{(s^2 + 1)^2} \right) = \mathcal{L}^{-1}(G(s)G(s)) = \sin t * \sin t.
\]
Compute the convolution using the product trig. identity
\[
2 \sin A \sin B = \cos(A - B) - \cos(A + B)
\]
to simplify the integral:
\[
\int_0^t \sin(t - t_0) \sin t_0 \, dt_0 = \frac{1}{2} \int_0^t \cos(t - 2t_0) - \cos(t) \, dt_0 =
\]
\[
= -\frac{1}{4} \sin(t - 2t_0) \bigg|_{t_0=0}^{t_0=t} - \frac{1}{2} t \cos t = \frac{1}{2} \sin t - \frac{1}{2} t \cos t.
\]

**R7 (solution).** Similar to the box example from the notes. Observe that with \( z(t) = it \) we can convert this to
\[
\int_{-\infty}^{\infty} \frac{2e^{2x}}{e^{4x} + 1} \, dx.
\]
This is the same example from the notes, with a few different constants.

**R8 (solution).** The process is the same as in the sine transform example from the lecture notes; the only difference is a minus sign and the integrals are simplified using \( f(x) = f(-x) \) instead of \( f(x) = -f(-x) \).
Note that if starting with a \([0, \infty)\) transform, you’d use the cosine transform.

**R9 (solution).** The shift rule is important here; it is similar to the transport equation example (and not like the heat equation). This is an example in the textbook; p603-605.

**Remark:** In fact the answer is essentially the same as the transport equation despite the equation being different. This is because the wave equation can be thought of as
\[
(\partial_t + c\partial_x)(\partial_t - c\partial_x)u = 0
\]
which is a composition of transport equations with speeds \( \pm c \). For this problem, solutions can only propagate to the right, so we only get the right moving \((x - ct)\) wave, just like \( u_t + cu_x = 0 \).